UNIVERSITAT DE LES
ILLES BALEARS

Master Thesis

The effect of temporal disorder on Complex Systems: Temporal Griffiths Phases.

Author:
Ricardo Martínez García.

Supervisor:
Cristóbal López Sánchez.
and Federico Vázquez.

September 20, 2011

The dissertation of Ricardo Martínez García. is approved:

Chair Date

Date

Date

University of Balearic Islands, The University of the Balearic Islands

Summer 2011

The effect of temporal disorder on Complex Systems: Temporal Griffiths Phases.

by

Ricardo Martínez García.

B.S. (Universidad de la Laguna) 2010

THESIS

Presented to

The University of the Balearic Islands

in Partial Fulfillment

of the Requirements

for the Degree of

MASTER OF SCIENCE

September 2011

La vida es aquello que te va sucediendo
mientras te empeñas en hacer otros planes

John Lennon.

Acknowledgements.

Me gustaría agradecer en primer lugar a mis dos directores Cristóbal López y Federico Vázquez todo lo que he aprendido durante este año, y lo que queda.

A todos los compañeros del IFISC. A Luis por estar siempre preparado para echarme una manos, o dos. A Pablo, por uruguayo. A Àngel, adoptado entre nosotros, por enseñarme el mundo de la pesca. A Neus, por aguantarme además todo el año en el Máster. Y a todos los demás por crear un ambiente en el que es fácil pasar tantas horas. Gracias también a Edu, Rubén y Antònia por tener siempre todo listo.

Tampoco me gustaría olvidarme de la gente que he conocido fuera del IFISC y que han hecho que salir de Tenerife haya valido la pena. A Luca, por poner en peligro este trabajo hasta el último momento con sus continuas visitas a Calviá, a mis compañeros de piso. Gracias también a Kike, Luis, Gloria, Adri, Alba, Alfonso...por hacerme sentir en casa aún estando a 2129 km de ella.

Finalmente, y quizá lo más importante, gracias a mis padres, que han hecho lo posible y un poco más para que sea lo que soy, y a mi hermano, por darme caña siempre y porque no se lo digo nunca.

Por último gracias a ti, por interesarte un poco en mi trabajo y haber cogido este Trabajo Fin de Máster, aunque haya sido sólo para leer los agradecimientos. Espero, que, si sigues, disfrutes con la lectura.

Contents

1

Introduction.

In Science, *Complexity* is a meaningful word. It is not used just to refer to complicated and not yet explained phenomena. The Science of Complexity emphasises the interactions between components of a system. Is the triumph of the collective behavior over the single particles. It stresses that components, most often, are heterogeneous and evolve in time, and it is concerned with the emergent properties at systems level originating from the microscopic interactions scales.

The object of research of this branch of Science are the *Complex Systems*. They consist of a huge number of constituents whose nonlinear interactions give rise to emergent hierarchial structures. The tools necessary to study all this phenomenology are provided by Statistical Mechanics, a mathematical formalism which can be applied to many different sciences including economics, population biology or sociology. It is, hence, a powerful Cross-Disciplinary methodology for the study of emergent phenomena at a macroscopic level caused by the many interactions taking place at microscopic scales. It provides a framework which makes possible to encapsulate the huge number of microscopic degrees of freedom of a complex system into just a few at a macroscopic scale.

One the most active fields of research in Complex Systems is the modeling of ecosystems [1, 2, 3], where many species interact giving rise to different global behaviors. It is interesting to know the conditions needed to have a coexistence or dominance of a single species. Taking into account spatial degrees of freedom the formation of patterns may be studied too.

Ecological systems are strongly influenced by external enviromental conditions like climate, floods or droughts. That is, the geophysical variability. The question of how external variability affects the diversity, robustness and evolution of complex systems remains poorly understood. It looks sensible, then, to look for a way of understanding its phenomenology where external time dependent conditions play a central role. In this sense, some models for Savannas have been proposed [4].

Furthermore, the mean lifetime of populations under environmental noise has been studied by Leigh [5] and Kamenev [6]. They showed that this time changes from exponential to power-law in the size of the system because of the effect of the noise. Besides, it has been also shown that spatial disorder changes the behavior of the system making new phases appear [7, 8]. Inspired by these works, a fundamental question arises, what motivated the present Masther Thesis: **does the temporal disorder induce new phases analogous to the ones induced by spatial disorder?**

Looking for an answer, the behavior of models with many particles which interact is studied under the effect of temporal disorder. The memory has the following structure. Firstly the mathematical tools and physical concepts used in the work will be introduced (Chapter 2), showing the different ways to describe this kind of problems. Secondly, the concept of disorder will be explained focusing our attention on the case of spatial disorder and trying to introduce the Griffiths Phases (Chapter 3). Lastly, the aim of the last chapter will be to explain what we understand as temporal disorder (Chapter 4) as well as the original research on its effects on some phase transitions. From there, the concept of Temporal Griffiths Phases will emerge naturally. Conclusions and a summary will close this text (Chapter 5).

2

Concepts and tools.

In this Master Thesis, the behavior of some many interacting particle models near the critical point and under time varying conditions is investigated. Therefore, this first chapter introduces both physical concepts and mathematical tools needed to understand further work. First of all how to describe systems used in the work is explained [9]. Secondly the dynamics followed by the system is described [10], stressing the mathematical tools used at different levels [11, 12, 13]. The chapter ends with a brief discussion on phenomena taking place near a critical point both in equilibrium [14, 15, 16] and nonequilibrium systems [17, 18, 19].

2.1 Systems of many interacting particles.

Systems composed by many particles[1] which interact in a non-linear way and evolve stochastically following some *updating rule* are one of the most active topics of research in Statistical Mechanics. This is due to its validity to model a wide range of processes in many fields of Science, going from condensed matter to biological or social systems.

The simplest models are going to be considered here, where each particle is placed in a node of a lattice and may be in one of two possible states. Each particle only interacts with its neighborhood, larger the longer is the range of the interaction. Therefore, the spatial variables are important and one works with an *extended system*.

An analogy with magnetism is often done and the particles are considered just as binary variables called *spins*. At each time step, the state of the system is

[1] Richer terminology may be also found in the literature depending on the discipline. They are called Agent-Based models in Sociology, Computer Science or Game Theory, and Individual-Based models in Ecology and Biology.

determined by giving the value of every spin, that is, the *configuration* of the system, also called *microstate*. The microscopic updating rule defines the dynamics of the model. Equivalently, the transition rates from one configuration to the next one can be writen obeying the update rule, they are

$$\omega_{c\to c'} \geq 0. \tag{2.1}$$

The description is closed by specifying the initial configuration and the geometry and the interactions among nodes in the lattice.

In summary, the way of working with interacting particle models at this level of description (that we call microscopic) is:

- Build a lattice with the desired geometry and interactions. From now on only regular lattices will be considered (Fig. 2.1).
- Specify an initial condition.
- Evolve in time, in accordance with updating rules, the initial condition. It is done using computational tools via Monte Carlo algorithms.
- Measure, either during the dynamics or once the final state is reached, the quantities one is interested in.

Once at this point, one can already guess the main advantage of these models: they make possible to perform numerical experiments on the collective behavior of complex systems in a simple way simulating its dynamics.

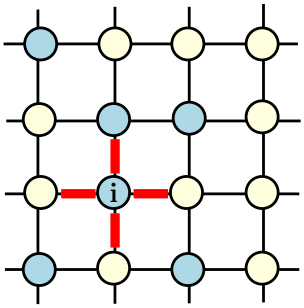

Figure 2.1. Typical two dimensions regular lattice used in this work. Here particles are placed in the nodes with two possible states represented by blue and yellow circles. The range of the interactions is only until nearest neighbors (NN), as it is represented by fat red lines in the i^{th} node.

2.2 Description of the system.

Apart from the strongly numerical approach introduced in Section 2.1, it is quite useful to make an analytical treatment which provides much deeper information

about the phenomenology taking place in the system. However, systems one usually works with have a huge number of particles. Writing equations of motion for every particle, using either classical or quantum mechanics, and solving them becomes impossible[2]. Unavoidably, the number of degrees of freedom in the system has to be decreased somehow. But, how can it be done?

First of all, one has to choose the relevant scales of interest of the problem. Then, macroscopic variables to describe the coarse-grained state of the system as a whole must be defined. They are called *collective variables* and fluctuate in a stochastic way. Some examples are density of particles, magnetization in spin systems or population of species in ecological models. They usually indicate how ordered the system is, so are also called *order parameter*. It is important to note that at this level (we call it macroscopic) some information is lost, but still a lot of information can be obtained. By giving the value of the collective variable at a certain time t one does not really know anything about the configuration of the system. Actually, each value of the collective variable defines a *macrostate* in the system, which may come from lots of different microstates (Fig. 2.2). On the other hand, it is possible to predict the macroscopic behavior from the microscopic dynamics.

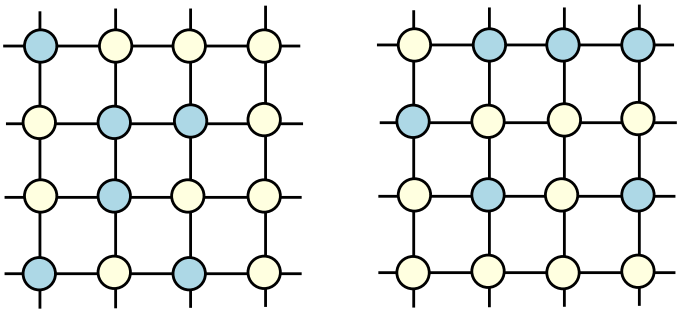

Figure 2.2. Example of a macrostate coming from two different microstates. Taking the density of particles in the blue state as the collective variable in both cases it is $\rho_{blue} = 3/8$. Nevertheless it is clear that the configurations of the system (microstates) are different.

The mathematical background used in this micro-macro connection are provided by Statistical Mechanics. Besides, as most of these models are out of the equilibrium, all the thermodynamic relations cannot be derived as it is usually done in Equilibrium Statistical Physics from Hamiltonian functions [26]. Now, time plays a relevant role and analytical techniques to treat non-equilibrium problems must be used, master equations, the Fokker-Planck equation or Langevin equations (Section 2.3). However, it is still hard to obtain analytical results using this via, and a stronger approximation is usually done. It is called the *mean-field* and provides a good first approach to the problem and qualitatively good results.

[2]Note that a a macroscopic system has about 10^{23} particles. It means that one should solve a system of coupled differential equations of that dimension

2.2.1 The Mean-field approach.

The main problem one faces with when trying to make a mean-field (MF) treatment of a problem is the huge number of different approximations that can be found in the literature under this name. That's why it is advisable to explain what is going to be done here.

Following [20, 21], the mean-field approximation in interacting particle models is obtained considering the limit of a weak and long range interaction among the constituents. Each particle in the lattice is allowed to interact with all the other in the system. Therefore, from the microscopic point of view *fully connected networks* are considered under this approach (lattice where all the nodes are neighbors), neglecting spatial dependence.

From a more physical point of view, the mean-field shall be understood as considering every particle in interaction with a global field coming from all the other particles in the lattice. The lack of a direct interaction between the neighbors causes one of the most important properties of the mean-field: *each particle in the lattice is uncorrelated with the other.* That is

$$\langle s_i s_j \rangle = \langle s_i \rangle \langle s_j \rangle. \tag{2.2}$$

In a strict sense, fluctuations of the collective variable around its mean value have to be neglected for having a complete mean-field approximation. Then, the stochastic differential equation (SDE) describing the evolution of the order parameter becomes deterministic, a rate equation.

The mean-field approximation does not provide quantitative good results in systems where local interactions are relevant but it becomes better as far as the dimensionality of the problem, d, increases. In the limit of infinite dimension they agree with the exact solution of the problem. A *critical dimension*, d_c, different for every model can be defined. It determines a treshold in the system in the sense that, for dimensions higher than the critical, mean-field results become the exact solution of the model.

2.3 Description of the dynamics.

Models whose state at a given time depends only on its previous configuration will be studied here. They follow a *Markovian dynamics*.

Due to the stochastic nature of interacting particle models, the description of the state of a system is done using equations of motion (master equation, the

Fokker-Planck equation) for the probability distributions of having either a configuration (microscopic description) or a value of the collective variable (macroscopic). It is also done by a differential equation for the studied collective variable, where the fluctuations are considered in a stochastic term (Langevin equation). These possibilities will be explained here.

2.3.1 The Master Equation.

One of the ways of describing the dynamics of the system will be to specify the probability of finding a given configuration c at a given time t, $P_c(t)$. The equation for this magnitude may be writen as

$$P_c(t+\Delta t) = \left(1 - \sum_{c'} \omega_{c\to c'}\Delta t\right) P_c(t) + \sum_{c'} \omega_{c'\to c}\Delta t P_{c'}(t), \tag{2.3}$$

where c' denotes the set of configurations of the system that can be reached from c.

Consequently, the right hand side of (2.3) represents all the possible previous states of the system, and the probability of going from there to the current one. The first term is the probability of already being in c and nothing happening, while the second one represents a gain due to the probability of being in c' and leaving it towards c.

Working with (2.3), in the limit of small enough time step, $\Delta t \to dt$, it becomes an equation of motion for the probability of finding each configuration c,

$$\frac{P_c(t+\Delta t) - P_c(t)}{\Delta t} \equiv \frac{\partial P_c(t)}{\partial t} = \sum_{c'} \omega_{c'\to c} P_{c'}(t) - \sum_{c'} \omega_{c\to c'} P_c(t). \tag{2.4}$$

Gain and loss terms in (2.4) balance each other, so the probability distribution remains normalized. Besides, as the coefficients $\omega_{c\to c'}$ are rates rather than probabilities they have units of $[time]^{-1}$ and may be greater than one.

This kind of equations are often hard to solve because they are a set of several, many times infinite, coupled first order ordinary differential equations. The most common techniques to solve them analytically involve transformations as the generating function, Fourier or Laplace transform [22]. Only in few simple cases the solution for all time t can be found, and often one has to be satisfied with knowing the first moments [23] of the distribution or just run numerical simulations [24]. Since the solution of master equations is not the goal of this memory, a deeper analysis will not be done here, but may be found in the references.

2.3.2 Fokker-Planck Equation.

The description in terms of a master equation can be understood as the system jumping from one microstate to another with given rates. However, as it was asserted before, it is not easy to obtain the probability distribution from this point of view. That's why sometimes the jumping process is approximated by a diffusion one, which leads to simpler equations. It is also important to remark that this approximation is better the smaller the jumps are, so the master equation becomes a Fokker-Planck equation in the limit of infinitely small jumps.

There are many ways of going to a Fokker-Planck from a master equation. It is explained here following the Kramers-Moyal expansion [11] in spite of its lack of mathematical rigor. In following chapters much more rigorous expansions of the master equation will be done.

First of all, take a master equation where jumps are so small that the set of possible states can be considered as a continuous. Then (2.4) becomes

$$\frac{\partial P_c(t)}{\partial t} = \int \left(\omega_{c'\to c} P_{c'}(t) - \omega_{c\to c'} P_c(t)\right) dc'. \tag{2.5}$$

Lets consider now in (2.5) the transition rates as a function of the size of the jump, r, and of the starting point, c,

$$\omega_{c\to c'} = \omega(c; r) \qquad r = c' - c. \tag{2.6}$$

Then, the master equation (2.5) becomes

$$\frac{\partial P_c(t)}{\partial t} = \int \omega(c-r; r) P_{c-r}(t) dr - P_c(t) \int \omega(c; -r) dr. \tag{2.7}$$

Two basic assumptions are done now. First of all one has to consider that only small jumps occur. That is, $\omega(c'; r)$ is a sharply peaked function of r but varies smoothly with c'. Mathematically, it means that

$$\omega(c'; r) \approx 0 \qquad \text{for } |r| > \delta, \tag{2.8}$$

$$\omega(c' + \Delta c; r) \approx \omega(c'; r) \qquad \text{for } |\Delta c| < \delta. \tag{2.9}$$

On the other hand, the solution, $P_c(t)$, varies slowly with c as it is in (2.9), making possible to perform a Taylor expansion up to second order in (2.5) to deal with the shift from c to $c - r$:

$$\begin{aligned} \frac{\partial P_c(t)}{\partial t} &= \int \omega(c; r) P_c(t) dr - \int r \frac{\partial}{\partial c} [\omega(c; r) P_c(t)] dr + \\ &+ \frac{1}{2} \int r^2 \frac{\partial^2}{\partial c^2} [\omega(c; r) P_c(t)] dr - \int \omega(c; -r) P_c(t) dr. \end{aligned} \tag{2.10}$$

First and fourth terms in (2.10) cancel each other. Writing the other two by defining the jump moments

$$\alpha_\nu(c) = \int_{-\infty}^{+\infty} r^\nu \omega(c; r) dr, \tag{2.11}$$

the result is

$$\frac{\partial P_c(t)}{\partial t} = -\frac{\partial}{\partial c}[\alpha_1(c) P_c(t)] + \frac{1}{2}\frac{\partial^2}{\partial c^2}[\alpha_2(c) P_c(t)]. \tag{2.12}$$

This is the Fokker Planck equation, derived here from the master equation. However, as it was said before, this derivation is not the most suitable because an explanation of what small parameter is taken is not given. Besides, there are many processes in which it fails. This is the case of systems with jump size ± 1 or some small integer, whereas typical sizes of the variable may be large, e.g., the number of molecules in a chemical reaction or the position of a random walker on a long lattice. In those cases expansions where the small parameter is explicitly taken are much more appropiate (Chapter 4).

Anycase, independently of the way used to write it, the Fokker-Planck equation describes exactly a large class of very interesting stochastic processes in which the system has a continuous sample path. That is, when the variable that describes the macrostate of the system can be written as a continuous function of time. This function will also be random, obeying a probabilistic law given by the solution of the Fokker-Planck equation (2.12), $P_c(t)$.

This leads to consider the description of the dynamics of the system in some direct probabilistic way, so that there will be a stochastic differential equation for the path of the system. This procedure was started by Langevin and is discussed now, in Section 2.3.3.

2.3.3 Langevin Equation.

Assuming that some information on the microscopic state of the system will be lost, sometimes interacting particle systems are described giving the time evolution equation of the collective variable already defined (order parameter), the Langevin equation. Lets call it M consistently with Section 2.4.1. This order parameter, as far as it describes the stochastic dynamics of the system, has a random time evolution.

Under a mean-field approximation, where the spatial degrees of freedom do not play a relevant role, the Langevin equation that turns up most often can be written in the form

$$\frac{dM}{dt} = f(M, t) + g(M, t)\eta(t), \tag{2.13}$$

where, $f(M,t)$ and $g(M,t)$ are known functions and $\eta(t)$ is the rapidly fluctuating random term. It will be required $<\eta(t)>$, the average over stochastic realizations, to be zero since any nonzero mean can be absorbed into the definition of $f(M,t)$. An idealization of a term like $\eta(t)$ must be that for $t \neq t'$, $\eta(t)$ and $\eta(t')$ are statistically independent (white noise from now on),

$$\langle \eta(t)\eta(t') \rangle = \Gamma \delta(t-t'), \tag{2.14}$$

where Γ gives the strength of the noise term.

Actually, the differential equation (2.13) does not exist, though the corresponding integral equation,

$$M(t) - M(0) = \int_0^t f[M(s),s]ds + \int_0^t g[M(s),s]\eta(s)ds, \tag{2.15}$$

can be defined consistently from the interpretation of the integral of the white noise as the Wiener process $W(t)$ (see [11, 12] for deeper argumentation):

$$dW(t) \equiv W(t+dt) - W(t) = \eta(t)dt. \tag{2.16}$$

Hence

$$M(t) - M(0) = \int_0^t f[M(s),s]ds + \int_0^t g[M(s),s]dW(s), \tag{2.17}$$

where the second integral can be defined like a kind of Riemann integral with respect to a sample function $W(t)$ (Appendix A).

However, this lack of mathematical rigor leads to some problems of interpretation. For instance, when in a stochastic differential equation the white noise appears multiplicatively ($g(M,t) \neq 0$), $M(t)$ won't be, in general, a continuous function of time. In those cases, there are ambiguities in some mathematical expressions, so giving a sense to the undefined expressions will constitute one of the main goals when integrating a Langevin equation. The most widely used interpretations are those of Itô and Stratonovich (Section A.1). The Itô integral is mathematically and technically the most satisfactory, but it is not always the most natural choice physically. The Stratonovich integral is more suitable, for instance, when $\eta(t)$ is a real noise with finite correlation time, where the vanishing correlation time wants to be taken (Section A.4). Both choices will be developed, and one or the other will be used depending on the situation.

Langevin equations are also valid when it comes to study systems further than a mean-field approach. When the space is important a new term appears in the equation in order to include the diffusion in the system [25]. As far as spatial degrees of freedom have to be considered, the collective variable is a continuous field,

$\phi(\vec{x},t)$. It does not depend only on time but also on the position, $\vec{x}$. The Langevin equation becomes

$$\frac{\partial\phi(\vec{x},t)}{\partial t} = f(\phi(\vec{x},t),t) + \nabla^2 M(\vec{x},t) + g(\phi(\vec{x},t),t)\eta(\vec{x},t), \tag{2.18}$$

which is a Stochastic Partial Differential equation. This approach is quite useful for spatially extended systems or to study pattern formation in the system.

From the Langevin to a Fokker-Planck equation.

It is also interesting to consider how are the descriptions in terms of a Fokker-Planck and by a SDE related each other. Starting from a Fokker-Planck equation as (2.12) but for the probability distribution of the order parameter M

$$\frac{\partial P(M,t)}{\partial t} = -\frac{\partial}{\partial M}\alpha_1(M)P(M,t) + \frac{1}{2}\frac{\partial^2}{\partial M^2}\alpha_2(M)P(M,t). \tag{2.19}$$

it is easy to write a Langevin equation of the type (2.13)

$$\frac{dM}{dt} = f(M,t) + g(M,t)\eta(t). \tag{2.20}$$

In the Itô interpretation coefficients in both equations are related according to

$$f(M,t) = \alpha_1(M,t), \tag{2.21}$$
$$g(M,t) = \sqrt{\alpha_2(M,t)}. \tag{2.22}$$

It makes sensible to call in the Fokker-Planck equation the *drift term* to the first one, because it leads to the deterministic part of the Langevin equation, and the *diffusion term* to the second, because the stochastic term in the Langevin equation emerges from it.

In the Stratonovich scheme an additional drift appears,

$$\frac{dM}{dt} = f(M,t) + \frac{1}{2}g(M,t)\partial_M g(M,t) + g(M,t)\eta(t), \tag{2.23}$$

with $\eta(t)$ being a white, Gaussian and zero mean noise.

Finally, it is interesting to conclude by saying that in the thermodynamic limit, $N \to \infty$ and $V \to \infty$, but with N/V finite, as the diffusion term vanishes typically as $N^{-1/2}$, a deterministic treatment may be done (Section 2.2.1, rigurous mean-field).

Sometimes, this way is walked on the inverse sense. That is, one may start with the deterministic equation and use heuristic arguments to obtain the stochastic

description. Working on this way means adding some Langevin forces to the deterministic equation, getting a stochastic differential equation equivalent to a Fokker Planck equation for properly chosen forces.

As a summarize, Figure 2.3 presents the different stages of treating a system of many interacting particles.

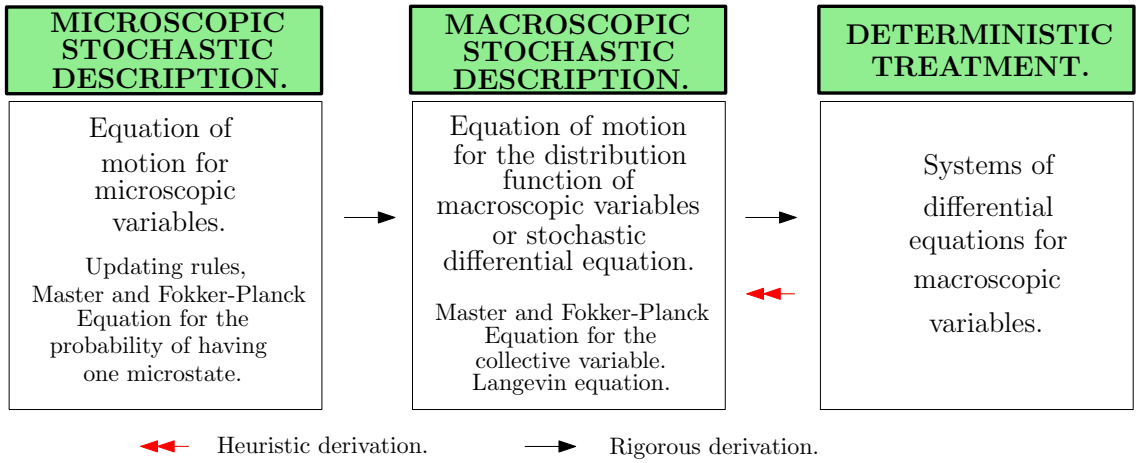

Figure 2.3: Three stages of treating a complex system.

2.4 Equilibrium dynamics.

Once the way of describing systems of many interacting particles has been briefly introduced, it is time to explain how is the dynamics they follow. Two situations are mainly possible. On the one hand, when they are in the thermodynamic equilibrium a complete formalism exists in such a way that the thermodynamic macroscopic properties of the system can be derived from its microscopic characteristics [26]. On the other hand, much less is known when the system is far from equilibrium (Section 2.5), as it happens in most of the systems around in Nature.

The only condition that the dynamical rules of a system must verify for being in equilibrium is the so called *detailed balance*

$$P_{eq}(c)\omega_{c\to c'} = P_{eq}(c')\omega_{c'\to c}, \tag{2.24}$$

that means that the probability currents between a pair of microstates of the system c and c' have to cancel each other (Fig.2.4 [Left]).

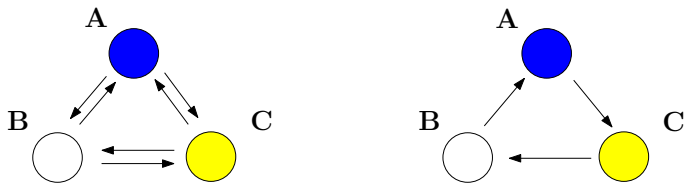

Figure 2.4. [10]. Detailed balance (Left) and non equilibrium steady states (Right). The figure shows a system with three microstates labeled by A, B, C. In both cases the stationary distribution function is $P_s(A) = P_s(B) = P_s(C) = 1/3$. On the left side the transitions occur at equal rates in all the directions, so the effective probablity currents vanish and the dynamics obeys detailed balance. On the right figure transitions occur only clockwise leading to non-vanishing probability currents. Therefore, the stationary state of the system is out of equilibrium.

2.4.1 Critical phenomena.

The term critical phenomena is used to refer to the thermodynamic properties of systems near a critical point of a phase transition. These situations are specially interesting because the microscopic degrees of freedom of the system behave collectively over large scales and a rich phenomenology emerges. The critical point will be one point in the phase diagram where differences between phases disappear, the characteristic length scale of the system diverges and various measurable quantities present singularities. It is characterized by a critical value of the control parameter, the parameter governing the dynamics of the system.

In this Master Thesis, the interest is focussed on order-disorder and active-absorbing phase transitions. These kind of transitions appear in many systems in Nature. Some examples are the paramagnetic-ferromagnetic transition, the extinction of species in ecology, opinion formation models, language coexistence models...

The transition can occur in two different ways. One possibility is that the order parameter of the system changes continuosly, exhibiting the system some kind of partial order for a range of values of the control parameter. These are the so called *second-order phase transitions* (Figure 2.5). The other possibility is that the change may be discontinuous from a total order in the system to a disordered state just crossing the critical point. These are the *first-order phase transitions*.

Magnetic systems are the typical physical example presenting order-disorder second-order phase transition, so the discussion will be restricted to them, without loss of generality, from now on. The most studied model is the Ising Model (Figure 2.5). In this case, the control parameter governing the dynamics of the system is the

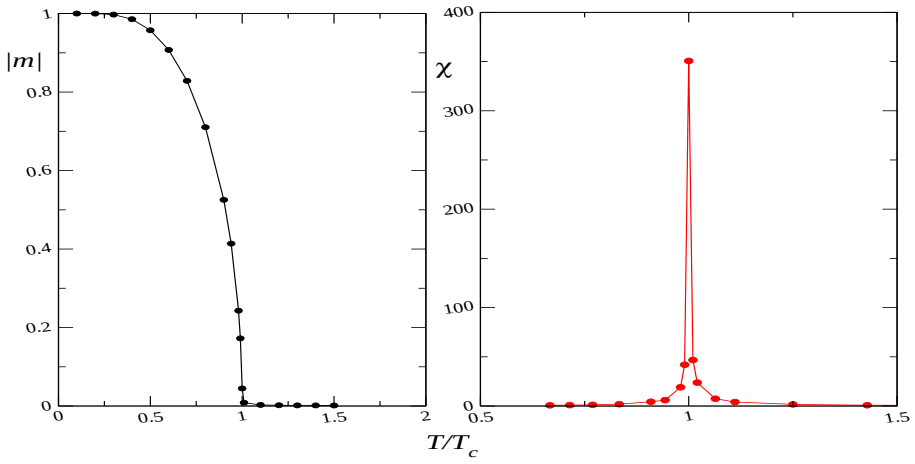

Figure 2.5. Monte Carlo simulations run on a fully connected network with 10^6 spins of an Ising Model following Glauber dynamics. Averages taken over 10^4 realizations. Left: order-disorder second-order phase transition using the magnetization per spin $m = M/N$, with N being the total number of spins in the system, as the order parameter and the temperature as the control parameter. Right: divergence in the susceptibility. Instead of it, because of finite-size effects a peak is observed at the critical temperature.

temperature[3], T, and the scalar order parameter, M, is called the magnetization,

$$M = \sum_{i}^{N} S_i, \tag{2.25}$$

where S_i are the spin variables.

A change in the order parameter in magnetic systems implies a work done in the system. It is given by

$$dW = hdM, \tag{2.26}$$

defining the conjugate field h, which is the external magnetic field. Using h and the temperature as independent thermodynamic variables, all thermodynamic functions can be derived from the Gibbs free energy $G(h, T)$:

$$\text{Magnetization:} \quad M = -\frac{\partial G}{\partial h}, \tag{2.27}$$

$$\text{Susceptibility:} \quad \chi = \frac{1}{V}\frac{\partial M}{\partial h}, \tag{2.28}$$

$$\text{Internal Energy:} \quad U = G - T\frac{\partial G}{\partial T}, \tag{2.29}$$

$$\text{Heat capacity:} \quad C = T^2\frac{\partial^2 G}{\partial T^2}, \tag{2.30}$$

where V is the volume of the system. Note also that from equations (2.27-2.30) the origin of the nomenclature used for calling the phase transitions is clear. First-order

[3]It is usually taken also som adimensional parameter proportional to T^{-1}. It changes the placement of the ordered and disordered phases.

ones show the discontinuity in the first order derivatives of the thermodynamical potential $G(h,T)$, while second-order do in the second derivatives, the susceptibility, χ, and the heat capacity, C.

Singularities in the measured quantities near the critical point are characterized by few critical exponents, so the singular part behaves as a power law. These exponents are shown to depend only on important properties of the model such as symmetries, dimensionality and conservation rules and not on particular microscopic characteristics. This is the concept of *universality* [4] and makes possible a classification of phase transitions in different families or *universality classes* [19, 27, 28] according to the value of the exponents. That's why **in this thesis a simple model, which captures the essential features of its universality class is taken as a representative of each "family". Then, introducing temporal disorder in the control parameter the change in the critical behavior is studied.**

The other main characteristic of critical phenomena, the *scale invariance*, comes from this power law behavior of some quantities. A scale-invariant system is one in which all thermodynamic functions are homogeneous functions verifying

$$f(bx) \sim b^p f(x), \tag{2.31}$$

when the unit of length is increased by a factor b. This is a general property of power law functions, which would transform as

$$f(x) \sim x^{-p} \qquad f(x/b) \sim b^p f(x). \tag{2.32}$$

Therefore, *systems near a critical point are scale-invariant.*

Ising universality class.

It is one of the most relevant universality classes in equilibrium statistical mechanics since the Ising model, one of the most deeply studied models, belongs to it. The energy of each spin configuration of the system is given by

$$\mathcal{H} = -J \sum_{<i,j>} S_i S_j - h \sum_i S_i, \tag{2.33}$$

where J is the coupling constant, h is the magnetic field and S_i is the spin variable at site i, $S_i = \pm 1$. This model undergoes a continuous phase transition from an ordered phase to a disordered one as far as the control parameter crosses its critical value b_c and if the magnetic field vanishes, $h = 0$.

[4] The concept of universality was first introduced by experimentalists in order to describe the observation that several apparently unrelated physical systems are sometimes characterized by the same type of singular behavior near the transition

The most important symmetry in $\mathcal{H}$ is the invariancy under a $S \to -S$ transformation when $h = 0$, so the ground state have twofold degeneracy. Many generalization of this model can be done, every model with the same symmetries in its "Hamiltonian"[5], $\mathcal{H}$, than the Ising model will belong to its universality class, sharing the same critical behavior and, then, the value of the critical exponents.

2.5 Non-equilibrium dynamics.

A physical system is said to be out of the equilibrium if the microscopic processes violate the detailed balance as shown in Figure 2.4 [Right]. That system has three microstates and jumps only clockwise, so the probability current does not vanish. Although the stationary distribution ($P_s(A) = P_s(B) = P_s(C) = 1/3$) corresponds to an equilibrium one, the system is out of it just because the detailed balance is not fulfilled. In general terms, every system subjected to external currents, such as supply of energy, is expected to be out of equilibrium.

Even though a well established formalism does not exist for working with nonequilibrium systems, some of them close to it can be treated as being in equilibrium. However, much richer phenomenology appears when working with systems far from it as will be shown during the main part of this memory.

2.5.1 Non-equilibrium phase transitions.

Critical phenomena and particularly phase transitions in systems far from equilibrium have been one of the most active fields of study in Statistical Physics last few years. Some relevant concepts for this Thesis are:

Absorbing states.

An important class of non-equilibrium phase transitions occurs in models with so-called absorbing states. They are configurations of the system where the dynamics is trapped, there are no fluctuations, and the system cannot escape from them. Since systems can reach but not leave absorbing states, they are the most clear example of a violation of detailed balance. They are present almost everywhere in nature (forest fires, ecological diffusion, propagation of epidemics...) [17, 29, 30, 31].

[5] $\mathcal{H}$ is not a classical Hamiltonian in the strict sense. Dynamical equations cannot be derived from it.

Systems with only one absorbing state are the most simple and thus the first in being studied, but those with two symmetric absorbing states are also very important because they present a big variety of phase transitions. Depending on the value of the parameters, they reach one of the absorbing states in different ways. All the diversity of phase transitions appearing in these systems will be the leading thread of this memory, trying to study the influence of temporal disorder on them.

Non-equilibrium universality classes.

Despite the intense research, a classification in terms of critical exponents as it is done in systems relaxing to equilibrium is not as well established [32]. Even though the concept of universality also seems to work out of equilibrium, the universality classes are expected to be more diverse as they involve time as an extra degree of freedom, and are governed by symmetry properties of the evolution dynamics. Two are the most interesting here, since they will be investigated in later work.

a) Directed Percolation (DP).

It is probably the one evolving the simplest phenomenology. Usual spreading models (forest fires, diseases, flows in porous media...) belong to it, sharing the same critical exponents. The representative of DP in this thesis is the Contact Process (CP) [33], which will be explained in Section 4.3. The main conditions a model must verify to belong to DP [34, 35] are:

1. Presence of a continuous phase transition from a fluctuating phase to a unique absorbing state.
2. The transition is characterized by a positive one-component order parameter.
3. The dynamic rule involves only short-ranged processes.
4. The system has no special attributes such as additional symmetries.

b) Generalized Voter (GV).

The main characteristics of this class are:

1. Presence of two symmetric competing absorbing states.
2. First order phase transition. It is not continuous anymore.
3. The dynamics of models undergoing phase transitions in the GV universality class is governed by *interfacial noise*. It means that the activity is only in the frontiers between domains of particles in different states. This is the opposite case of the Ising model, where due to thermal fluctuations there may be activity in a domain, *bulk noise*.

It will be explained in Section 4.1 that the GV transition is sometimes understood as an order-disorder transition (Ising) and an active-absorbing transition (DP) taking place so close that they are not distinguishable. Therefore, the GV transition consists on a simultaneous symmetry breaking, and a DP absorbing transition that for some reasons might be separated when the neighborhood considered in the dynamics is modified (introduction of bulk noise).

3

Spatial disorder.

In disordered systems, knowing whether and how the critical behavior changes when introducing a small amount of uncorrelated impurity has become one of the central questions to answer. The one considered here will be the *dilution*, that is, the absence of spins in some places of the lattice. This kind of disorder leads to models with quenched disorder i.e. depending on the spatial variables but which does not evolve in time [32,8]. It is not considered here the presence of an external field, only intrinsic disorder.

This theoretical question has a strong experimental motivation since real systems in Nature contain impurities rather than being pure models. In fact, this is the reason why a phase transition presented in a huge variety of systems, as the Directed Percolation, was observed experimentally only three years ago. The presence of noise tends to change its behavior, so measures in the laboratory did not agree with theoretical predictions.

The most interesting phenomenology takes place near and at the critical point, so many questions come to mind. Some of them are:

- Will the phase transition remain only at one point in presence of spatial disorder or does the critical point splits somehow?

- If so, will the critical behavior change quantitatively, giving new universality classes with new critical exponents, or even qualitatively with new non-power law scalings at criticality?

- Will only the transition be influenced by the noise or also its vicinity?

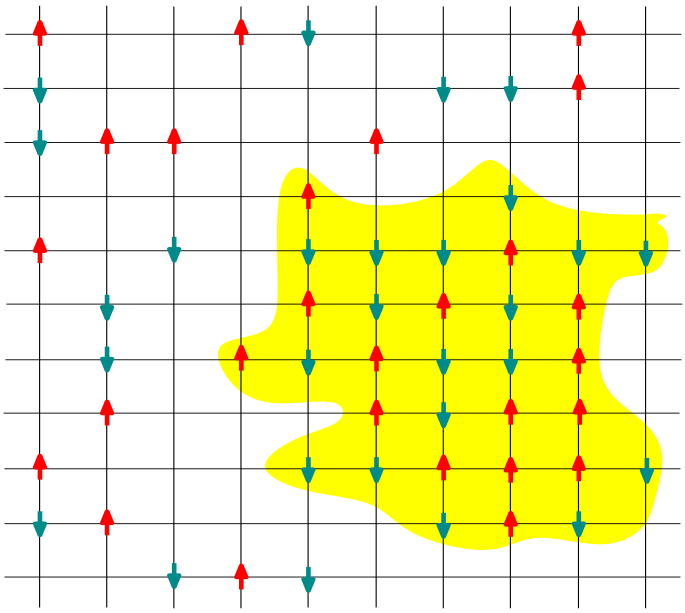

Figure 3.1: Rare region in a diluted spin interacting model.

3.1 Rare regions. The Griffiths Phase.

The dilution reduces the tendency towards magnetic long-range order in the system. Therefore, the critical value of the control parameter for the pure model (without noise), $b_{c,pure}$, moves into the ordered phase, $b_{c,q}$ (Fig. 3.2).

On the other hand, in an infinite system, as happens in the thermodynamic limit, it is possible to find a region without vacancies of an arbitrary size, pure regions. When the value of the control parameter, typically temperature, is between $b_{c,pure}$ and $b_{c,q}$, though the whole system is in the disordered phase these pure regions can exhibit some local order, which means a local value of the magnetization which does not vanish. These pure spatial regions are called *rare regions* (Figure 3.1) and the probability of finding them decreases exponentially with its size V_{RR} and the impurity concentration, p. Furthermore, the dynamics in these regions is very slow since a coherent change is needed in order to flip them.

The interval between $b_{c,pure}$ and $b_{c,q}$, in the disordered phase, (Figure 3.2) is the so-called *Griffiths Phase*, because it was Griffiths the first who showed the possible existence of a singularity in the free energy in this region [7]. They are of relevance in condensed matter physics, because of the presence of non-magnetic impurities in magnetic systems [36], as well as in other contexts like complex networks [37] or quantum systems [8]. Its main characteristic are the generic divergences of thermodynamic magnitudes, such as the susceptibility, as a consequence of the singularity in the free energy, and an anomalously slow relaxation to zero of the order parameter. Besides, other time-dependent quantities also relax specially slow, mainly as a

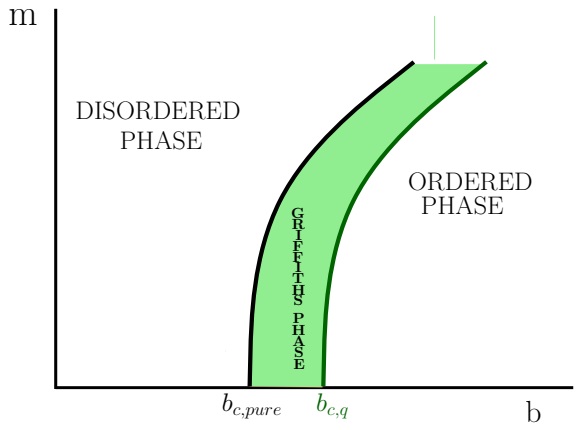

Figure 3.2: The Griffiths Phase.

power law or a stretched exponential, in contrast with the fast decay typical of pure systems, usually exponential.

Lets call p_{RR} the probability of finding a rare region in a system with impurity concentration, p. It is exponentially small in its d-dimensional volume V_{RR} and in p. Up to constant factors, it is

$$p_{RR} \propto \mathrm{e}^{-pV_{RR}}, \tag{3.1}$$

which means that rare regions are not perturbative. The key for understanding the importance of large rare regions is to know how the contribution of a single region to observable quantities scales with its size. Three different cases are possible:

1. A single rare region cannot undergo a phase transition and the contribution of an isolated rare region grows, at most, as a power law with its size. It means that it cannot overcome the exponential scaling in the probability of finding it, so its effect is not important on the system. The critical behavior will be still a power-law, as happens in pure systems.

2. The rare regions cannot undergo phase transitions independently yet, but their contribution increases exponentially with the size L_{RR}[1]. the exponential decay of probability of finding a rare region with a given size (3.1) can be overcome. When the control parameter yields between $b_{c,q}$ and $b_{c,pure}$, inside the Griffiths Region, rare regions in the system dominate the long-time behavior and any finite region decays exponentially. For the density ρ it is

$$\rho(t) \propto \int dL_{RR} L_{RR}^{d} \mathrm{e}^{-t/t_R} p_{RR}, \tag{3.2}$$

[1]It is related to the d-dimensional volume by $V_{RR} = L_{RR}^{d_{RR}}$

where t_R is the characteristic decay time of a region of size L_{RR} and d is dimensionality of the system. In average, since a coordinate fluctuation is needed in order to flip the whole region, the activity there decays as

$$t_R \propto \mathrm{e}^{aV_{RR}}, \tag{3.3}$$

with a being a constant which vanishes at $b = b_{c,pure}$ and increases while decreasing T [38]. V_{RR} as said in (3.1) is the volume of the rare regions. A saddle-point approach can be done in the integral in (3.2), expanding the sharp function around its maximum. It leads to a power law decay in ρ

$$\rho(t) \sim T^{-p_{RR}/a}. \tag{3.4}$$

It is important to note that owing to the behavior of a with the control parameter b, the algebraic decay of this magnitude is with a continuously varying exponent.

3. The rare regions can undergo the phase transition so the dynamics of the locally ordered regions freezes. The global phase transition is destroyed, since different spatial parts of the system order at different values of the control parameter.

4

Temporal disorder.

The treatment of some problems in physics, chemistry, or ecology needs parameters to be disordered in time rather than in space[1]. This is for instance the case of ecosystem modeling, where the geophysical variability plays an important role. It is clear that weather conditions are relevant in the dynamics of such systems and they change depending on the season of the year [4].

In this sense, the works of Leigh [5] and Kamenev et al. [6] motivated the research presented in this chapter. They showed that, in one-variable (mean-field) models of stochastic populations, environmental noise changes the system mean lifetime, defined as the time needed to reach the absorbing state, from exponential to a power-law in the size of the system.

With this background, Vázquez et al. investigated the properties of a specific model with an absorbing state under temporal disorder [39]. The chosen model was the Contact Process, presenting a phase transition from an active phase to an absorbing one belonging to the Directed Percolation universality class. Through the study of the susceptibility and the mean lifetime, phenomenological similarities with the Griffiths Phases were found motivating the concept of Temporal Griffiths Phases.

In this Master Thesis these results are extended [40] to the more general framework provided by systems with two symmetric absorbing states, which present three types of phase transitions that define three universality classes. **With this aim, we took a simple model as representative of each class. Then, temporal disorder was introduced in the control parameter, b and the behavior of the model near the critical point was studied. Differences with the pure model, where b is constant, were observed. The question is, how does temporal disorder affect the dynamics?**

[1] Of course spatiotemporal disorder might be required in many other cases, but it won't be considered here.

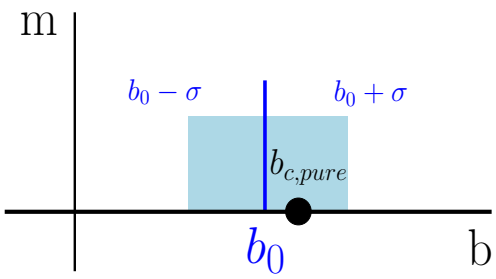

Figure 4.1. Possible values of the control parameter. The systems shifts between tending to order and disorder.

Consider the pure case, where the control parameter is $b = constant$. Depending on the value of b the system will tend to an ordered or disordered phase, exhibiting critical behavior only at one value of $b = b_{c,pure}$. What is done here is to introduce temporal fluctuations in this parameter around a mean value, b_0. This is performed sorting a random number from a uniform distribution from $-\sigma$ to σ after a time $\mathcal{T}$, and adding it to the mean value b_0. It makes the control parameter to be an stochastic function of time,

$$b(t) = b_0 + \sigma\xi(t), \tag{4.1}$$

where $\xi(t)$ is a random number in the interval $[-1, 1]$.

As b_0 will be close to the critical point of the pure model, $b_{c,pure}$, **choosing a proper value of σ one forces the system to instantaneously shifts between the tendencies to be ordered or disordered.**

What one observes is a subphase in the active phase where the mean lifetime scales algebraically with continuously varying exponent and there are generic divergences in the susceptibility. This is the so called *Temporal Griffiths Phase* because of its phenomenological similarities with the Griffiths Phase appearing in the presence of spatial disorder.

4.1 Systems with two absorbing states.

To begin with, lets introduce models with two absorbing states, characterized by the absence of fluctuations, trapping the dynamics of the system. To obtain two absorbing states, the model should present some particular properties, that can be summarized as:

- Inactive sites, of one of the two absorbing states, spontaneously appear in active domains.

- The boundaries between inactive and active domains fluctuate in a way that domains grow.
- Boundaries between domains of different absorbing states behave as active sites, meaning that they can produce activity in both domains. This is specially important in order to have behavior different from DP. If this rule is not satisfied, neighboring domains of different absorbing states behave as a single absorbing block.

It is interesting to study the critical phenomenology present on these kind of models. The first thought in this sense was made by Dornic et al. [41] who extrapolating numerical simulations in $d = 2$ conjectured that transitions with Z_2 symmetry should belong to the GV class if there are no bulk fluctuations in the modeling. However, later on, Droz, Ferreira and Lipowski [29] showed that in some generalized voter models with interactions up to third nearest-neighbors ($Z = 12$ neighbors in regular two dimensions lattices) such a transition may not be direct, but split into an Ising, where symmetry is broken, followed by a DP transition to the chosen absorbing state. One or the other scenario depends on the parameters in the model.

To explain this, it is worth to write down a generic macroscopic equation for the model. It is possible because, in spite of having each model different dynamical rules, many of them seem to share the same macroscopic behavior, such as criticality and coarsening. There are two main ways of achieving such an equation, the first one followed by Vázquez and López [30] connects the microscopic dynamics with the macroscopic evolution in square lattices. The second is to derive it from general properties in the model; this is the way chosen by Al Hammal et al [31].

Here, the later will be followed writing the Langevin equation and studying its behavior in the parameter space. The considerations made in the derivation are:

- The equation has to be symmetric under reversal of the field ($\phi \to -\phi$).
- The field takes values between two absorbing barriers, set at $\phi = \pm 1$.
- The absorbing barriers must be similar to those of the Langevin equation for DP since in two dimensions the GV critical point can split into an Ising and a DP point. Therefore the square root of the distance to each of the absorbing barriers ($\sqrt{1-\phi^2}$) must appear as a multiplicative factor of the noise.
- In order to represent the possibility of Ising-like spontaneous symmetry breaking, the deterministic part of the equation needs a minimal number of polynomial terms with odd powers of ϕ.
- At least two free parameters are needed to cover the splitting in the critical point scenario.

This leads to a Langevin equation which in its simplest form can be written as

$$\frac{\partial\phi(\vec{x},t)}{\partial t} = (a\phi(\vec{x},t) - c\phi^3(\vec{x},t))(1-\phi^2(\vec{x},t)) + D\nabla^2\phi(\vec{x},t) + \Gamma\sqrt{1-\phi^2(\vec{x},t)}\eta(\vec{x},t), \tag{4.2}$$

where $\eta(\vec{x},t)$ is a white Gaussian noise of zero mean and delta correlation both in space and time. a, c and D are real coefficients and Γ is also a real parameter that modulates the noise strength.

At mean-field level, neglecting the noise term and the spatial dependence in $\phi(\vec{x},t)$, Eq. (4.2) can be studied. Taking into account only its first term the problem can be mapped to the motion of a non mass particle under the action of some potential

$$\frac{d\phi}{dt} = -V'(\phi), \tag{4.3}$$

with

$$V(\phi) = -\frac{a}{2}\phi^2 + \frac{a+c}{4}\phi^4 - \frac{c}{6}\phi^6. \tag{4.4}$$

This potential can be studied in the parameter space (see Figures 4.2 and 4.3, where the parameter space is sketched and shapes of the potentials for different values of the parameters plotted):

- The voter model behavior is recovered in a single point of the parameter space, $a = 0$, $c = 0$, where the potential vanishes and the Langevin equation would be the one proposed for the original voter model [42]

$$\frac{\partial\phi}{\partial t} = D\nabla^2\phi + \Gamma\sqrt{1-\phi^2}\eta(t). \tag{4.5}$$

- If $c > 0$ the system presents Ising and DP transitions.
  - While $a < 0$, $\phi = 0$ is stable, so it is the stationary state.
  - At $a = 0$, $\phi = 0$ becomes unstable and one expects the symmetry to be broken and an Ising transition to take place.
  - With $0 < a < c$ growing, minima in the potential move towards $\phi = \pm 1$.
  - Once $a \geq c$ both minima reach the absorbing barriers, so the stable states are $\phi = \pm 1$. There is a DP transition when fluctuations are considered.

- If $c < 0$ there is a unique GV transtion. Though the term in ϕ^4 appears in the potential it is not stabilizing, so there is a first order transition. Minima in the potential change abruptly from $\phi = 0$ to $\phi = \pm 1$ when crossing from $a < 0$ to $a > 0$. It also presents additional maxima when $a < 0$ which may or not lie in the interval $[-1, 1]$, but they tend to $\phi = 0$ when $a = 0$.

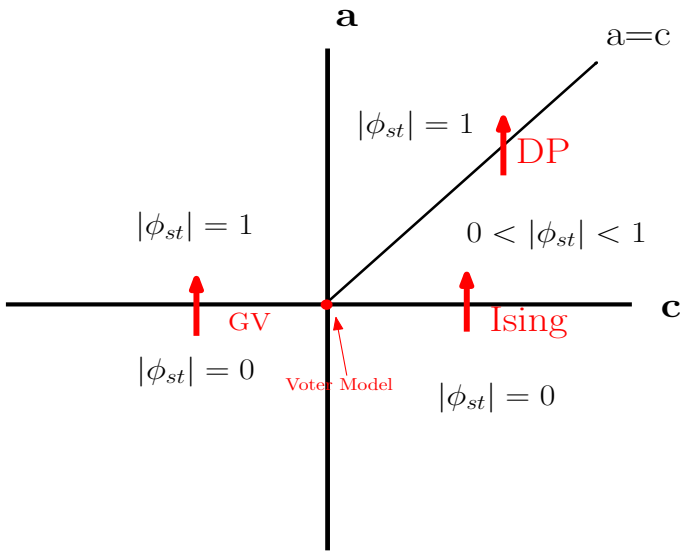

Figure 4.2. Parameter space of the equation (4.2). The classification of the different phase transitions in systems with two symmetric absorbing is shown depending on the values taken by a and c.

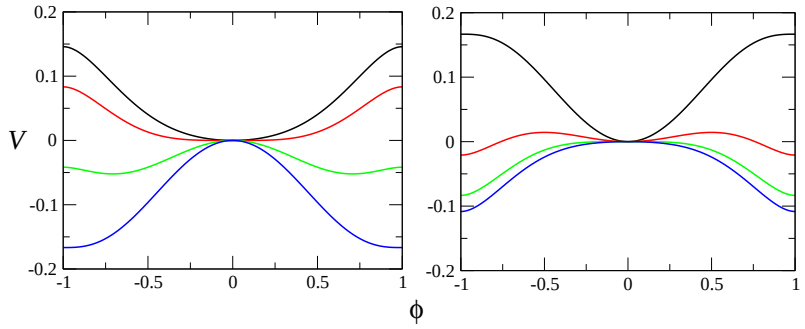

Figure 4.3. Plot of the potential for different values of the parameters. *Left:* $c = 1$. Ising and DP transition are taking place. From top to bottom, $a = -0.25, 0, 0.5, 1$. *Right:* $c = -1$. Unique GV transition. From top to bottom, $a = -1, -0.25, 0, 0.1$.

In what follows, all these transitions are studied under the effect of temporal disorder in the control parameter. To this aim, we consider three different models that are known to exhibit each of the three types of transitions and add temporal disorder in their dynamics.

4.2 Macroscopic Mean field description of the models: General Approach.

As it has been already explained, a description of interacting particles in terms of its microscopic dynamics is quite difficult and often unsolvable analytically. To gain insight in the problem a mean field approach is always interesting. In this section a Langevin equation at a mean field level will be derived in the general case where the particles are taken as binary variables, $S_i = \pm 1$. First of all, a collective variable, the magnetization per spin, (Section 2.4.1), is defined

$$m = \frac{\sum_{i=1}^{N} S_i}{N}, \tag{4.6}$$

where N is the total number of spins. In a single spin-flip event, the possible changes in m are $\Delta m = \pm 2/N$. In the $N \to \infty$ limit they become infinitesimal, allowing one to take m as a continuous but stochastic time dependent variable.

Assuming that spins can only flip one by one, the probability $P(m,t)$ of having magnetization m at time t, obeys the master equation

$$\begin{aligned} P(m,t+\Delta t) &= \omega_+\left(m-\frac{2}{N}\right)P\left(m-\frac{2}{N},t\right)+\omega_-\left(m+\frac{2}{N}\right)P\left(m+\frac{2}{N},t\right)+ \\ &+ \left[1-\omega_-(m)-\omega_+(m)\right]P(m,t), \end{aligned} \tag{4.7}$$

where $\omega_\pm(m)$ are the transition probabilities and the subscripts indicate if m increases $(+)$ or decreases $(-)$. $\Delta t = 1/N$ is the time step in which a spin is chosen at random.

Now, as the the size of the system N is high enough and changes both in magnetization and time are small, Taylor expansions can be done. The first one is developed up to second order and around m in the right-hand side of (4.7). In the left-hand side, the expansion is also done but around t and to first order. Arranging terms properly, it is,

$$\frac{\partial P(m,t)}{\partial t} = -\frac{\partial}{\partial m}\left[f(m)P(m,t)\right]+\frac{1}{2}\frac{\partial^2}{\partial m^2}\left[g_1(m)P(m,t)\right], \tag{4.8}$$

which is the Fokker-Planck equation with drift and diffusion terms given, respectively, by

$$f(m) = 2\left[\omega_+(m) - \omega_-(m)\right], \tag{4.9}$$

$$g_1(m) = \frac{4\left[\omega_+(m) + \omega_-(m)\right]}{N}. \tag{4.10}$$

In general, this equations of motion for the probability distribution,(4.8), is not easy to solve. It is convenient to write a single Langevin equation for the evolution of m itself. This is done in terms of stochastic differential equations. From (4.8), and working on the Itô scheme, because it comes from a discrete in time equation [43] (Appendix A), the resulting equation is

$$\frac{dm(t)}{dt} = f(m) + \sqrt{g_1(m)}\eta(t), \tag{4.11}$$

where $\eta(t)$ is a white Gaussian noise of zero-mean and correlation $\langle \eta(t)\eta(t')\rangle = \delta(t-t')$. It is important to remark that the diffusion term is proportional to $1/\sqrt{N}$, so it will vanish in the thermodynamic limit, leading to a deterministic equation.

The drift and diffusion coefficients (4.9) and (4.10) in the Langevin equation (4.11) depend not only on the magnetization, but also on the control parameter, which is going to become stochastic. The dependence comes from the functional form of the transition probabilities $\omega_-(m)$ and $\omega_+(m)$. The point now is how is this new stochastic effect on the microscopic dynamics reflected in the macroscopic description in terms of the Langevin equation. Lets call b the order parameter of the system, whose definition will change from one model to another. The time dependence, as explained in the beginning of the chapter, arises from doing

$$b(t) = b_0 + \sigma\xi(t), \tag{4.12}$$

that is, making the control parameter fluctuate around a value b_0. Fluctuations are modulated by σ and come from the function $\xi(t)$ (Fig. 4.4) whose autocorrelation function is not a delta function but it decays linearly to zero in a time τ. Therefore, the noise $\xi(t)$ can be considered as a colored noise with zero mean and a step-like two points correlation,

$$< \xi(t)\xi(t+\Delta t) >= \begin{cases} 1/3 & \text{for } \Delta t < \mathcal{T} \text{ and } n\mathcal{T} \leq t \leq (n+1)\mathcal{T} - \Delta t, \\ 0 & \text{for } \Delta t < \mathcal{T} \text{ and } (n+1)\mathcal{T} - \Delta t \leq t \leq (n+1)\mathcal{T}, \\ 0 & \text{for } \Delta t > \mathcal{T}. \end{cases} \tag{4.13}$$

The key of this implementation is to take the distribution broad enough (by choosing a proper value of σ) to make the system instantaneously shift between the tendencies to be active ($b(t) > b_{c,pure}$) or absorbing ($b(t) < b_{c,pure}$).

With this new term, the model presents two kinds of fluctuations. The intrinsinc ones coming from the microscopic dynamics, and those coming from the

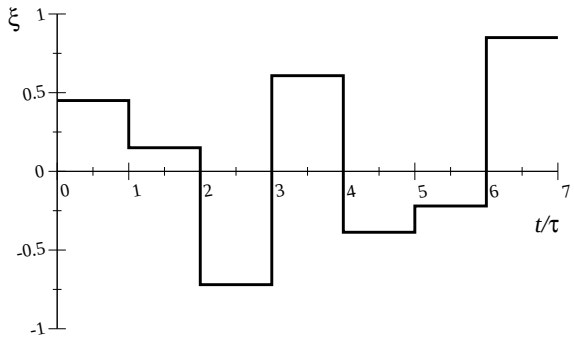

Figure 4.4: Typical realization of $\xi(t)$ which changes the value of $b(t)$ each time interval $\mathcal{T}$.

stochasticity in the parameter $b(t)$. The aim at this point is to split them in different terms of the Langevin equation. To this purpose, one should expand the drift and diffusion terms in (4.11) in such a way that now the order parameter appears only in the coefficients of polynomial functions. Working on this way and retaining only linear terms in the noise, one finally arrives to

$$\frac{dm(t)}{dt} = f_0(m) + \sqrt{g_{1,0}(m)}\eta(t) + g_{2,0}(m)\xi(t), \tag{4.14}$$

where $f_0(m)$ and $g_{1,0}(m)$ are the same functions than in (4.11) but depending on b_0 instead of b, $\xi(t)$ is the step-like function characterized by (4.13) and shown in Fig. 4.4, and $g_{2,0}(m)$ is a polynomial function that might depend as well on b_0.

Given that $\xi(t)$ is an stochastic function with correlation time $\mathcal{T}$, (4.14) can be considered as a stochastic differential equation with both a white, $\eta(t)$, and a colored noise $\xi(t)$. Then, assuming that relaxation times in the dynamics are much larger than $\mathcal{T}$, the limit $\mathcal{T} \to 0$ can be taken and (4.14) transformed into a Langevin equation with an effective white noise whose squared amplitude is the sum of the squared amplitudes of the intrinsic and extrinsic noises [12]. It leads to

$$\dot{m} = f_0(m) + \sqrt{g_{1,0}^2(m) + K g_{2,0}^2(m)}\Xi(t), \tag{4.15}$$

where the dot means time derivative and $\Xi(t)$ is a white ($< \Xi(t)\Xi(t') >= \delta(t-t')$) Gaussian noise of zero mean. The constant K is the effective diffusion which appears when going from a colored to a white noise [12] and defined as $K \equiv \int_{-\infty}^{+\infty} \overline{\langle \xi(t)\xi(t+\Delta t)\rangle} d\Delta t$. As

$$< \xi(t)\xi(t+\Delta t) >= \begin{cases} 1/3(1-|\Delta t|/\mathcal{T}) & \text{for } \Delta t < \mathcal{T} \\ 0 & \text{for } \Delta t > \mathcal{T}, \end{cases} \tag{4.16}$$

it is $K = \mathcal{T}/3$, with $\mathcal{T} = 1$ because the change in $b(t)$ is each Monte Carlo time step in the implementation of the microscopic dynamics.

Once a general Langevin equation for a model with a stochastic time dependence in the control parameter has been derived, one will be able to work with different models and phase transitions belonging to different universality classes just taking the proper functional form for the microscopic transition probabilities. This is the main goal of this Master Thesis.

4.3 Directed Percolation

Some previous results published in [39] on the effect of a time dependent control parameter on a Directed Percolation phase transition are presented here. As a representative of this transition the Contact Process was taken. In this model [33], each site of a d-dimensional lattice can be either occupied $S_i = 1$ (active) or vacant $S_i = 0$. At each time step, an active site is randomly chosen and, with probability b, it converts into active a nearest neighboring site, provided it was empty. On the other hand, with probability $1 - b$ it is declared empty. Thus, b is the control parameter. Time, t, is then increased by $1/N(t)$ where $N(t)$ is the number of active sites. The "pure" Contact Process, ($b(t) \equiv cte$), is critical only at some value $b_{c,pure}(d)$ depending on the dimension ($b_{c,pure}(d = 2) \approx 0.767, b_{c,pure}(d = 3) \approx 0.622, b_{c,pure}(d \to \infty, MF) \approx 0.5$) and separating and active phase, where there are active sites in the stationary state, from an absorbing phase, without active sites in the steady state. The order parameter will be, then, the density of active sites

$$\rho = \frac{\sum_{i=1}^{N} S_i}{N}. \tag{4.17}$$

However, due to fluctuations any finite system always reaches the absorbing state in the steady state. Nevertheless the mean lifetime of the active phase grows exponentially fast with the system size N, $\tau(N) \sim \mathrm{e}^N$, in the "pure" case $b(t) \equiv cte$ and in the active phase making it stable in the thermodynamic limit. Instead, in the time dependent Contact Process, as the control parameter $b(t)$ can be adverse for a time interval, the mean lifetime is expected to be significantly reduced. The question is, does it still diverge for $N \to \infty$? That is, does a stable active phase exist?

With this aim, Vázquez et al provided some numerical and analytical results [39] which will be reproduced here in order to introduce the concept of Temporal Griffiths Phases.

In the scheme of Section 4.2 a Langevin equation (4.15) can be writen for this model. The only thing to remark is that the order parameter of the problem is not the magnetization per spin m, but the density of active sites in the system ρ. Now the jump in the order parameter is $\Delta\rho = \pm 1/N$. According to the microscopic

dynamics, the transition probabilities are

$$\begin{aligned} \omega_-(\rho) &= \rho(1-b), \\ \omega_+(\rho) &= b\rho(1-\rho), \end{aligned} \tag{4.18}$$

which allow to write the master equation of the problem. Perfoming an $1/N$ expansion, the Fokker-Planck equation is easily obtained, and in the Itô sense the Langevin equation can be written. The coefficients are

$$f_0(\rho) = a_0\rho - b_0\rho^2, \tag{4.19}$$

$$g_{1,0}(\rho) = \sqrt{\frac{\rho}{N}}, \tag{4.20}$$

$$g_{2,0}(\rho) = \sigma\rho, \tag{4.21}$$

with $a_0 = 2b_0 - 1$. Then, the mean-field Langevin equation is

$$\dot{\rho} = a_0\rho - b_0\rho^2 + \sqrt{\frac{\rho}{N} + K(\sigma\rho)^2}\Xi(t), \tag{4.22}$$

with $\Xi(t)$ being a white, zero mean, Gaussian noise and K the effective diffusion coefficient that appears from transforming the colored noise $\xi(t)$ into a white noise.

4.3.1 Numerical results.

As a first approach to the problem some numerical results are provided in dimensions $d = 1$, $d = 2$ and $d \to \infty$ (fully connected network, mean-field). The values of the order parameter $b(t)$ are extracted from a flat distribution $[b_0 - \sigma, b_0 + \sigma]$ each Monte Carlo step. Experiments where the initial value of the density of active sites is $\rho = 1$ were done, measuring the time, $\tau(N)$, taken by the system for going to the absorbing state $\rho = 0$. While in $d = 1$ the usual power law scaling at criticality is observed ($\tau \sim N^{1.55}$), for $d = 2$ and $d \to \infty$ it is $\tau \sim [\ln N]^\alpha$ with $\alpha(d = 2) = 5.18(5)$ and $\alpha(d \to \infty) = 2.66(5)$ if $\sigma = 0.4$. Besides, in $d = 2$ and $d \to \infty$, but again not in $d = 1$, there is a whole region in the active phase where the mean lifetime grows generically as a power law, with continuously varying exponent δ. Finally in the absorbing phase $\tau \sim \ln N$ in all dimensions. To sum up, while in $d = 1$ the time dependent model seems to behave as pure models do, in $d = 2$ and $d \to \infty$ one observes

- Logarithmic scaling at criticality.
- An extended region with algebraic scaling.

These two main properties of the mean lifetime, which regard the properties of Griffiths Phases, are going to characterize the presence of Temporal Griffiths Phases.

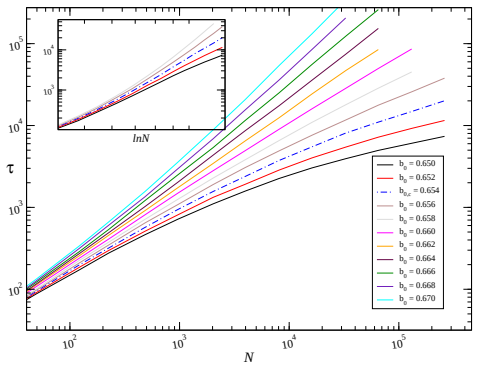

Figure 4.5. Main: log-log plot of the lifetime τ as a function of the system size N for the time dependent Contact Process in $d = 2$ with $\sigma = 0.4$ There is a finite region, $b_0 \in [0.656, 0.675]$ with generic algebraic scaling and continuously varying exponent. Inset: log-log plot of $\tau(N)$ vs lnN from a fit at criticality it is estimated $\tau \sim (\ln N)^{5.18(5)}$

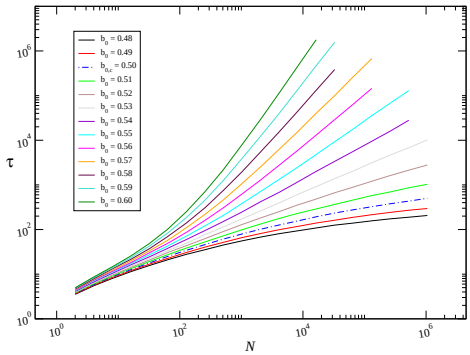

Figure 4.6. Main: log-log plot of the lifetime τ as a function of the system size N for the time dependent Contact Process in $d \to \infty$ with $\sigma = 0.4$.

Griffiths Phases also exhibit generic divergences in some measurable thermodynamics quantities. One of them is the magnetic susceptibility, defined as the response of the system to an external field,

$$\chi \sim \lim_{h \to 0} \frac{\partial \bar{\rho}}{\partial h}, \tag{4.23}$$

where the bar denotes average over independent realizations.

Looking for this properties also in Temporal Griffiths Phases, the Langevin equation (4.22) of the model was integrated both in MF ($d \to \infty$) and $d = 2$[2]. Results are shown in Figure 4.7, main and inset respectively. The main plot shows that the susceptibility diverges all along the TGP in mean-field, while the inset shows generic divergences for the $d = 2$ case. However, due to the impossibility of reaching low enough values of the external field h, one cannot confirm numerically if the divergence is real or just a transient effect.

4.3.2 Analytical results.

In the high dimensional limit (mean-field), given that at every time step the change on the order parameter is $\pm 1/N$, one can map its dynamics into a random walk in the interval $[0, 1]$, with jumps of length $\pm 1/N$ occuring with probabilities $\omega_{\pm}(\rho)$. Considering the mean lifetime, one can neglect the intrinsic noise ($g_{1,0}(\rho) = 0$)

[2]The Langevin equation for the finite dimension model also has the diffusive term.

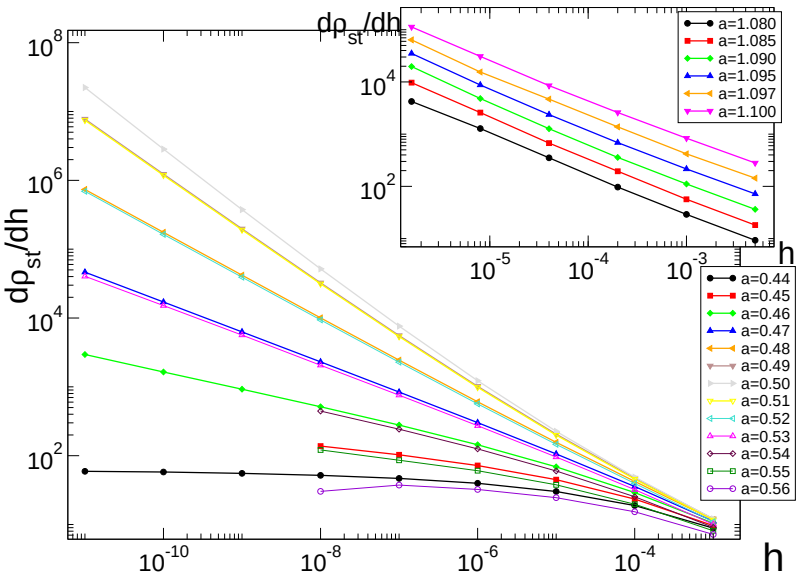

Figure 4.7. Log-log plot of the susceptibility χ as a function of the external field h, obtained integrating the Langevin equation of the Contact Process.

since $g_{1,0}(\rho) \ll (g_{2,0}(\rho)$ when N is high. The extinction time is then defined as the time needed for the active state to fluctuate and reach the vicinity of the absorbing state, $\rho = 1/N$. As the only fluctuations in the system come from taking the white noise limit of a colored one, the Langevin equation has to be interpreted in the Stratonovich sense [43], (Section A.4),

$$\dot{\rho} = (2b_0 - 1)\rho - b_0\rho^2 + \sigma\rho\Xi(t), \tag{4.24}$$

with $\Xi(t)$ being a white Gaussian noise of zero mean.

Now one defines the change of variables $z = \ln\rho$ working with the usual rules of calculus because the Stratonovich scheme is used. This transformation allows to convert the multiplicative noise in (4.24) into an additive one. The resulting Langevin equation is

$$\dot{z} = (2b_0 - 1) - b_0 e^z + \sigma\Xi(t) \tag{4.25}$$

which describes a random walk trapped in a potential $V(z) = -(2b_0 - 1)z + b_0 e^z$ and exhibits the three following regimes:

- Active phase, $b_0 > 1/2$. As the absorbing state is in the vecinity of $\rho = 1/N$, it is $z = -\ln N$. The time needed to reach the absorbing state is exponential in the height of the potential (Arrenhius' Law [12]).

  $$\tau(N) \sim \exp\left[\frac{V(-\ln N)}{\sigma^2/2}\right] \sim \exp\left[\frac{2a_0 \ln N}{\sigma^2}\right] \sim N^{2a_0/\sigma^2}, \tag{4.26}$$

  which means that $\tau(N)$ scales as a power law with continuously varying exponent. Hence, the active phase is stable when $N \to \infty$.

- Critical point, $b_0 = 1/2$. When z is small enough one has a free random walk, without potential barrier to be overcome. This movement covers a typical distance $\sqrt{\tau}$ in time τ. Equating this distance to $z = \ln N$, the mean lifetime scales logarithmically
$$\tau \sim [\ln N]^2. \tag{4.27}$$
- Absorbing phase $b_0 < 1/2$. The new variable z decays linearly in time and the time needed to reach the absorbing state scales as
$$\tau \sim \ln N. \tag{4.28}$$

Regarding the divergence of the susceptibility, one has to consider the Fokker-Planck equation associated to the Langevin equation (4.22) in the limit $N \to \infty$. Neglecting the intrinsic fluctuations forces one to use the Stratonovich interpretation as was done above. Finally it is possible to obtain its solution in the quasi-stationary state, from where one calculates the mean density of active sites as function of the external field and the magnetic susceptibility as its derivative. It shows a generic divergence
$$\chi \sim h^{-1+2|\tilde{a_0}-\tilde{a}_c|/\sigma^2} \tag{4.29}$$
with $\tilde{a_0} = 2b_0 - 1 + \sigma^2/2$ and $\tilde{a_c} = 2b_c - 1 + \sigma^2/2$; in an extended interval $\tilde{a} \in [0, \sigma^2]$ around the critical point [44].

4.4 Ising.

The original results of this Master Thesis [40] are presented in this Section and following. It will be studied here the transition belonging to the Ising class and appearing when the GV transition splits. This is done taking as a representative the usual Ising model following Glauber dynamics. The transition rates are
$$\Omega(S_i \to -S_i) = \frac{1}{2}\left[1 - S_i \tanh\left(\frac{b}{2d}\sum_{j\in\langle i\rangle} S_j\right)\right], \tag{4.30}$$
where the sum is over the $2d$ nearest neighbors of spin i on a d-dimensional hypercubic lattice and $b = J\beta$ is the control parameter. J is the coupling constant between spins and $\beta = 1/K_B T$ the Boltzmann's factor. In the following $J = 1$ is taken. The order parameter is the magnetization per spin.

4.4.1 Mean field.

In mean-field (fully connected network), the transition rates (4.30) become
$$\Omega(S_i \to -S_i) = \frac{1}{2}\left[1 - S_i \tanh\left(\frac{b\sum'_j S_j}{N-1}\right)\right]. \tag{4.31}$$

The prime on the sum denotes that it is over all spins in the system except for S_i. When $N \gg 1$, the term S_i can be included with negligible error, it leads to transition rates for the order parameter

$$\Omega_{\pm}(m) = \frac{1}{2}\left[1 \pm \tanh\left(bm\right)\right]. \tag{4.32}$$

Given this dynamics, the coefficients of the Langevin equation (4.15) are obtained. The probability for a flip $+ \to -$ taking place is the probablity of choosing an up spin, multiplied by the probablity of that spin to flip:

$$\omega_{-}(m) = \frac{1+m}{2}\Omega_{-}(m), \tag{4.33}$$

and equivalently

$$\omega_{+}(m) = \frac{1-m}{2}\Omega_{+}(m). \tag{4.34}$$

Expanding up to third order the hyperbolic tangent in the transition probabilities, the coefficients for the Langevin equation of the pure model, (4.11) are

$$\begin{aligned} f(m) &= am - cm^3, \\ g_1(m) &= \sqrt{\frac{1}{N}\left[1 - bm^2\right]}, \end{aligned} \tag{4.35}$$

with $a = b - 1$ and $c = b^3/3$.

Now the stochastic dependence in the control parameter has to be implemented as was explained in Section 4.2. In this case, it is done in detail to better understand the procedure. Consider the control parameter (4.12),

$$b(t) = b_0 + \sigma\xi(t), \tag{4.36}$$

writing it in the coefficients (4.35), they are

$$f(m) = (b_0 + \sigma\xi(t) - 1)m - \frac{(b_0 + \sigma\xi(t))^3 m^3}{3} \tag{4.37}$$

$$g_1(m) = \sqrt{\frac{1}{N}\left[1 - (b_0 + \sigma\xi(t))m^2\right]}. \tag{4.38}$$

The noise term in (4.38) is neglected because it will lead to higher order noise terms in the Langevin equation. In the drift term there are lineal terms in the noise which will give rise to the coefficient $g_{2,0}(m)$. Developing the cubic term in (4.37), neglecting nonlinear terms in the noise and rearranging, one finally achieves the coefficients of the complete Langevin equation for the time dependent Ising model

$$\begin{aligned} f_0(m) &= a_0 m - c_0 m^3 \\ g_{1,0}(m) &= \sqrt{\frac{1}{N}\left[1 - b_0 m^2\right]} \\ g_{2,0}(m) &= \sigma m(1 - b_0^2 m^2), \end{aligned} \tag{4.39}$$

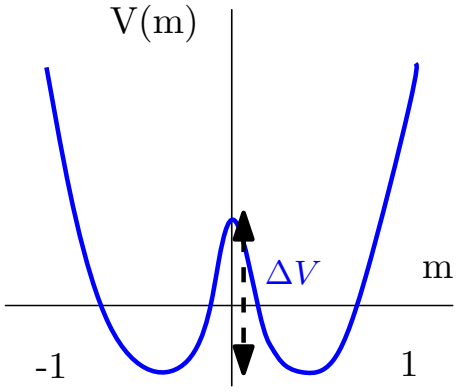

Figure 4.8. Sketch of the typical potential barrier ΔV to be overcome in the order phase. Minima correspon to the possible values of the magnetization in the steady state.

with $a_0 = b_0 - 1$ and $c_0 = b_0^3/3$.

The Langevin equation at a mean-field level is

$$\dot{m} = a_0 m - c_0 m^3 + \sqrt{\frac{1}{N}\left[1 - b_0 m^2\right] + K\left[\sigma m(1 - b_0^2 m^2)\right]^2}\,\Xi(t), \tag{4.40}$$

where $\Xi(t)$ is a delta correlated Gaussian noise of zero mean and K the effective diffusion appearing from taking the white noise limit in a colored one ($K = 1/3$).

Once this equation is obtained, numerical and analytical results can be provided.

Numerical results.

Since Ising Model does not present absorbing states, the mean escape time, τ[3], needs to be redefined. It will be taken as the time needed by the system to go from a totally ordered state ($m = \pm 1$) to a disordered one ($m = 0$). Note that, when the system is in the subcritical phase a potential barrier has to be overcome (Fig. 4.8). For this magnitude in the high dimensional limit both Monte Carlo simulations on a fully connected network and numerical integration of the Langevin equation (4.40) has been done. The results are shown in Fig. 4.9.

As it is characteristic in TGPs, the critical scaling changes from a power law when $\sigma = 0$ ($\tau \sim N$ for the Glauber Ising model) to $\tau \sim [\ln N]^\alpha$, with an estimated exponent $\alpha(FCN) = 2.81$ for $\sigma = 0.4$ (Inset Fig. 4.9). A broad region with power law continuously varying exponent scaling also appears in the ordered phase

[3]It is also denoted by τ but must not be confused with the mean lifetime (extinction time) defined in systems with absorbing state.

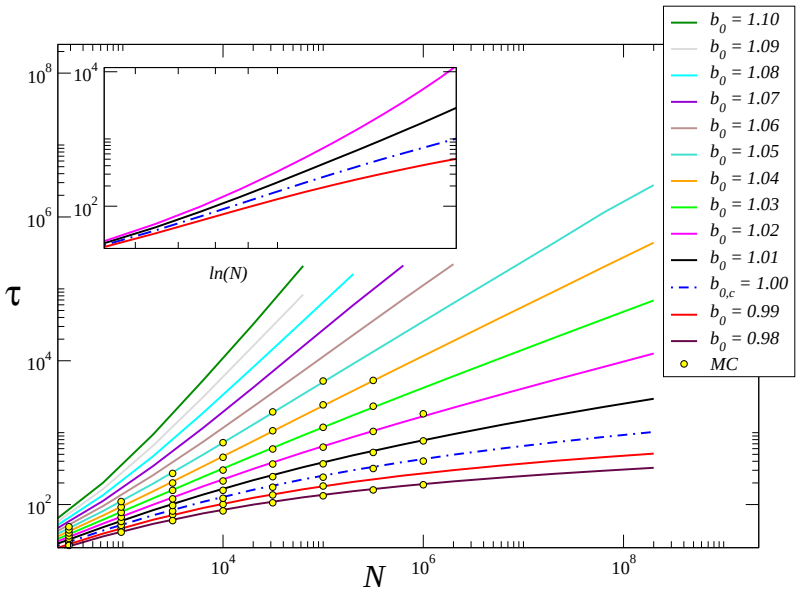

Figure 4.9. Main: Log-log plot of the escape time $\tau(N)$ for the Ising Model with Glauber dynamics in mean field. Monte Carlo simulations on a FCN (Dots) and numerical integration of the Langevin equation (4.40) (solid lines) with $\sigma = 0.4$. There is a region $b \in [1.01, 1.10]$ with generic algebraic scaling of $\tau(N)$ and continuously varying exponents. Inset: log-log plot of $\tau(N)$ vs. $\ln(N)$. It is estimated at criticality $\tau \sim (\ln N)^{2.81}$.

($\tau \sim N^{\delta}$), with $\delta \to 0$ when b_0 tends to its critical value in the time dependent model, $b_{0,c}$, coming from higher values. Both α and δ are not universal and depend, mainly, on the noise strength σ. Furthermore, no shift is observed in the critical point, remaining on the value that it takes in the clean model $b_{0,c} = b_{c,pure} = 1$.

Secondly, the susceptibility, $\chi(h)$, defined as the response function to an external field h in the limit of $h \to 0$, has been measured

$$\chi = \lim_{h \to 0} \frac{\partial \overline{m}}{\partial h}, \tag{4.41}$$

where the bar denotes the average of the magnetization over realizations. In the clean model, it diverges at the critical point, and remains constant for other values of the temperature. Numerical measures have been done integrating (4.40) with $\sigma = 0.1$, considering the presence of the external field h and going to the thermodynamic limit ($g_{1,0} = 0$). A generic divergence is observed (Fig. 4.10) in a broad region centered around $b_{0,c}$; $b_0 \in [b_{0,c} - \sigma^2/2, b_{0,c} + \sigma^2/2]$ with symmetric exponents around the critical point. These results agree with those obtained through Monte Carlo simulations on a FCN. In the case of finite dimension ($d = 2$) the impossibility of going to small enough values of the magnetic field via Monte Carlo simulations doesn't allow to conclude the existence of the divergence.

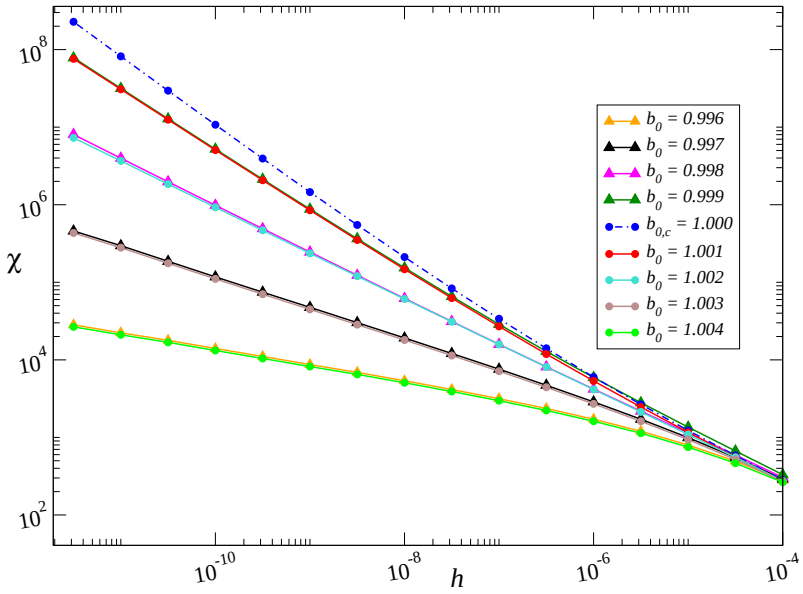

Figure 4.10. Main: Log-log plot of the susceptibilty as a function of the external field for different values of $b \in [b_{0,c} - \sigma^2/2, b_{0,c} + \sigma^2/2]$ with $\sigma = 0.1$ integrating Eq. (4.40) with a external field and in the thermodynamic limit . Generic divergences with symmetric exponents around the critical value $b_{0.c} = 1$ are observed.

Analytical results.

In order to support numerical results, analytical calculations are provided for the mean field case. Firstly, lets consider the case of the magnetic susceptibility. In the thermodynamic limit the Fokker-Planck equation has to be written. As the intrinsic noise of the model is neglected ($g_{1,0}(m) = 0$), the only stochastic part in the Langevin equation is due to the fluctuations in the order parameter. This noise comes from going to the white noise limit of a colored one so the Stratonovich interpretation has to be used. The resulting Fokker Planck equation is

$$\begin{aligned} \frac{\partial P(m,t)}{\partial t} &= -\frac{\partial}{\partial m}\left(f_0(m) + \frac{1}{2}g_{2,0}(m)g'_{2,0}(m)\right)P(m,t) + \\ &+ \frac{1}{2}\frac{\partial^2}{\partial m^2}g^2_{2,0}(m)P(m,t). \end{aligned} \tag{4.42}$$

In the stationary state $t \to \infty$ and neglecting non linear terms in $g_{2,0}(m)$, a first integration in m may be done. Taking the integration constant equal to zero, equation (4.42) is

$$\left[\left(a_0 + \frac{\sigma^2}{2}\right)m - c_0 m^3\right]P_{st}(m) + \frac{\sigma^2}{2}\frac{d}{dm}P_{st}(m) = 0. \tag{4.43}$$

Equation (4.43) can be analitically solved. Defining the change of variables

$$Q(m) \equiv m^2 P_{st}(m), \tag{4.44}$$

it becomes

$$\left(b_0 m - \frac{a_0 + \sigma^2/2}{m}\right) Q(m,t) = \frac{\sigma^2}{2}\frac{dQ(m)}{dm}, \tag{4.45}$$

which can be easily integrated. Its solution is

$$Q(m) \propto \exp\left(\frac{2}{\sigma^2}\tilde{a}\ln m - \frac{c_0 m^2}{\sigma^2}\right), \tag{4.46}$$

where $\tilde{a} = a_0 + \sigma^2/2$.

Finally, taking into account the change of variables (4.44), the stationary probability distribution function is

$$P_{st}(m) \propto \exp\left(-\frac{c_0 m^2}{\sigma^2}\right) m^{\frac{2\tilde{a}}{\sigma^2}}. \tag{4.47}$$

In order to study the behavior of the magnetic susceptibility, an external magnetic field has to be considered on the system. While in the microscopic transition rates it favors the spins to be parallel to it, in the macroscopic equation it will give rise to an additional term in the stationary probability distribution function,

$$P_{st}(m) \sim \exp\left[\frac{-c_0 m^2}{\sigma^2} - \frac{h}{\sigma^2 m}\right] m^{\frac{2\tilde{a}}{\sigma^2}-2}, \tag{4.48}$$

from where the mean magnetization per spin as a function of the field in the steady state can be obtanined $\overline{m}_{st}(h)$. The result (4.48) is analogous to that obtained in [39, 44, 45], where it is shown that the susceptibility derived from the probability distribution function (4.48) presents algebraic generic divergences as $h^{-\upsilon}$ with $\upsilon = (1 - 2|a - a_c|/\sigma^2)$ in a region given by $|a - a_c| < \sigma^2/2$. According to this result the power law in the critical point is $\upsilon = 1$ which is not good agreement with the numerical result $\upsilon = 0.88$. The critical exponent $\upsilon = 1$ seems to be reached in the limit of vanishing external field. Better fit is achieved for other values of b_0. For instance, an analytical value $\upsilon_{an} = 0.40$ for $b_0 = 1.003$ corresponds to a numerical one of $\upsilon_{num} = 0.39$ or $\upsilon_{num} = 0.59$ and $\upsilon_{an} = 0.60$ when it comes to $b_0 = 1.002$.

On the other hand, analytical results are also provided for the escape time. The Langevin equation for this time dependent Ising model (4.40) also describes the movement of a brownian particle with jump length $2/N$ and under the action of a time dependent potential (Fig. 4.3). Thus, obtaining the behavior of the time needed by the spin model to achieve a disordered state from an ordered one is equivalent to calculate the first passage time, T, of the particle through $m = 0$ starting from $m_i = 1$.

The first approximation to make the problem analitically solvable is to consider the limit $N \to \infty$. Then, the size of the system is introduced calculating the first

passage time through $m = 2/N$ instead of $m = 0$[4]. The resulting Langevin equation has to be interpreted in the Stratonovich sense because the only noise term comes from taking the white noise limit in a colored one. The resulting Fokker-Planck equation is again (4.42).

According to [12], the first time passage, T, is given by

$$\left[f_0(m) + \frac{D}{2} g_{2,0}(m) g'_{2,0}(m)\right] T'(m) + \frac{D}{2}\left[g^2_{2,0}(m)\right] T''(m) = -1, \tag{4.49}$$

whose solution is

$$T(m_i = 1) = 2 \int_{2/N}^{m_i=1} \frac{dy}{\psi(y)} \int_y^1 \frac{\psi(z)}{D g^2_{2,0}(z)} dz, \tag{4.50}$$

where

$$\psi(x) = \exp \int_{2/N}^{x} \frac{2 f_0(x') + D g_{2,0}(x') g'_{2,0}(x')}{D g^2_{2,0}(x')} dx'. \tag{4.51}$$

These integrals in (4.51) can be solved (Appendix B), and the obtained scaling for the mean escape time (4.50) is

$$T \sim \begin{cases} \frac{\ln N}{b_0 - 1} & \text{for } b_0 < 1 \\ \frac{3(\ln N)^2}{\mathcal{T}\sigma^2} & \text{for } b_0 = 1 \\ N^{\frac{6(b_0-1)}{\mathcal{T}\sigma^2}} & \text{for } b_0 > 1. \end{cases} \tag{4.52}$$

Results do not agree quantitative with those obtained numerically integrating the Langevin equation although the obtained exponents for the power law region also grow linearly with b_0. This lack of agreement is due to the approximation done when neglecting the $1/\sqrt{N}$ noise term which is shown to be important when $m \to 2/N$. Note that while the function $g_{2,0}(m)$ vanishes when $m \to 0$, $g_{1,0} \to 1/\sqrt{N}$. Numerical integrations of (4.40) taking $g_{1,0}(m) = 0$ show a better fit with the analytical exponent (Figure 4.11). Anycase, the qualitative behavior is obtained.

4.4.2 Finite dimension.

Numerical simulations have been also run on cubic regular lattices with nearest neighbor (NN) interactions for finite dimensions. When going to $d = 2$ a shift in the critical point is observed from $b_c = 0.441(1)$ in the clean model to $b_{0,c} = 0.605(1)$ when $\sigma = 0.4$. However, the power law scaling remains at criticality though with a different exponent than in the clean model ($\tau \sim N^{1.12}$ if $\sigma = 0.4$). Besides, the finite

[4]As m is not an actual continuous variable, one can think that when the distance to the final point is smaller than the length of the jump the system has already reached it

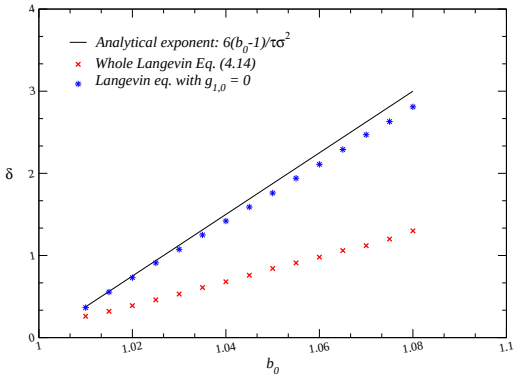

Figure 4.11. Comparison between analytical and numerical value for the power law exponents in the mean escape time. The line is the analitycal results, red crosses come from the numerical integration of the whole equation (4.40), while the blue stars are obtained taking $g_{1,0} = 0 (N \to \infty)$.

region with slow scaling does not appear and an exponential growth is observed in the ferromagnetic phase (Arrenhius' Law), as it happens when one takes $\sigma = 0$. On the other hand, for the $d = 3$ case a similar phase diagram to that sketched for the MF limit is observed. The critical exponent is estimated now around $\alpha = 5.13$ and the algebraic scaling appears again. However, a shift on the critical point is also observed. It moves to $b_{0,c} = 0.413(2)$ instead of the pure value $b_{c,pure} = 0.222(1)$ calculated in [46]. These results, shown in Fig. 4.13, agree with those achieved by Alonso and Muñoz in [47] in the sense that they also found that for the Ising Model a temporal disorder is relevant when $d \geq 3$. They provide numerical results on the scaling of the maximum in the peak of the susceptibility and conclude that a temperature homogeneous in space but random in time is a relevant perturbation when $d \geq 3$.

Regarding to the magnetic susceptibility, simulations do not allow to draw any conclusion because one cannot achieve low enough values of the magnetic field via Monte Carlo algorithms.

4.5 Generalized Voter.

The last transition explored in this work is that present in systems with two symmetric absorbing states when the $Z2$ symetry is broken and one of the absorbing states achieved at the same time. This is shown to belong to the Generalized Voter class. One model presenting this phase transition is the q-voter model when $q = 3$ as introduced in [48]. The microscopic dynamics of this nonlinear variety of the voter model consists on choosing one random spin S_i and flipping it with a given probability according to the state of the also randomly chosen q neighbors. If all of them are in the same state, S_i takes their state with probability 1 (remaining on its state if it was already parallel to the chosen neighbors). Otherwise it flips with

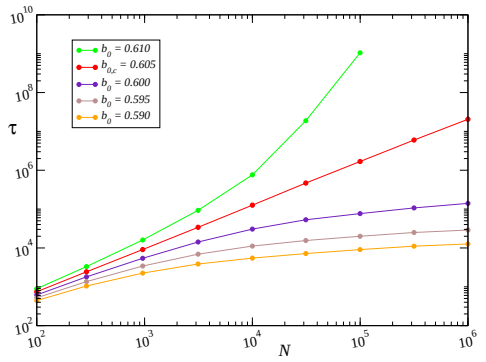

Figure 4.12. Log-log plot of the escape time $\tau(N)$ for the Ising Model with Glauber dynamics in $d = 2$. Monte Carlo simulations on a regular cubic lattice with $\sigma = 0.4$. The time scales as a power law at the critical point, exponentially in the ferromagnetic phase and as lnN in the paramagnetic phase, as it does in the pure model. A shift in the critical point is also observed from $b = 0.441(1)$ in the clean model to $b_{0,c} = 0.605(5)$.

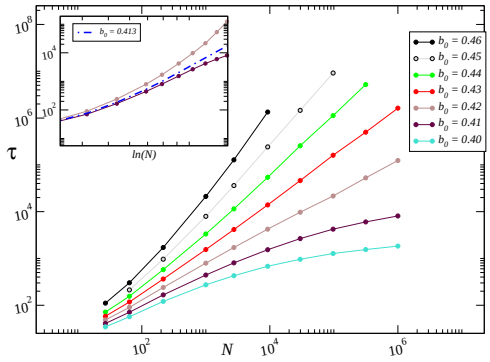

Figure 4.13. Main: Log-log plot of the escape time $\tau(N)$ for the Ising Model with Glauber dynamics in $d = 3$. Monte Carlo simulations on a regular cubic lattice with $\sigma = 0.4$. There is a region $b \in [0.42, 0.46]$ with generic algebraic scaling of $\tau(N)$ and continuously varying exponents. Inset: log-log plot of $\tau(N)$ vs. $\ln(N)$. It is estimated at criticality $\tau \sim (\ln N)^{5.13}$.

probability b. Thus, b is the control parameter of the model and the magnetization per spin the order parameter. Note the existence of 2 symmetric absorbing states when all the spins in the lattice are parallel.

The probability that a site flips as a function of the fraction of disagreeing neighbors x, and for $q = 3$, is

$$f(x, q) = x^3 + b[1 - x^3 - (1 - x)^3], \tag{4.53}$$

4.5.1 Mean field.

Given $f(x)$ according to (4.53), in mean-field, the transition probabilities $m \to m \pm 2/N$ are

$$\omega_+(m) = \frac{1-m}{2} f\left(\frac{1+m}{2}, q\right), \tag{4.54}$$

$$\omega_-(m) = \frac{1+m}{2} f\left(\frac{1-m}{2}, q\right), \tag{4.55}$$

then, the drift and diffusion coefficients of the Langevin equation (4.15) can be obtained working exactly in the same way as in the Ising model (Section 4.4). They are

$$\begin{aligned} f_0(m) &= \frac{1-3b_0}{2}m(1-m^2), \\ g_{1,0}(m) &= \sqrt{\frac{1}{N}\left(1-m^2\right)\left(1+6b_0+m^2\right)}, \\ g_{2,0}(m) &= \frac{3\sigma}{2}m(1-m^2). \end{aligned} \tag{4.56}$$

The complete Langevin equation is

$$\dot{m} = \frac{1-3b_0}{2}m(1-m^2) + \sqrt{\frac{1}{N}\left(1-m^2\right)\left(1+6b_0+m^2\right) + \frac{9\sigma^2}{4}Km(1-m^2)^2}\Xi(t). \tag{4.57}$$

Numerical Results.

Both numerical integration of 4.57 and Monte Carlo simulation of the microscopic stochastic dynamics have been done for measuring the mean lifetime in $d \to \infty$. This time is defined now as that taken by the system for going to one of the absorbing states from a random initial condition. It is important to note that a potential barrier has to be overcome again when $b > b_c$, while fast ordering is expected in the paramagnetic phase. Results in Fig. 4.14 show again a change in the critical behavior with an estimated critical exponent of $\alpha = 3.68$ for $\sigma = 0.3$, and a broad region of algebraic scaling with continuously varying non-universal exponent in the active phase. As expected in a mean-field approach, the critical point does not shift, and remains in the same value that in the pure model $b_{c,0} = 1/3$.

Analytical Results.

It is also possible to give analytical results. As it was done in Section 4.4, $g_{1,0} = 0$, but the studied brownian particle moves now from $m = 0$ to $m = \pm 1 \mp 2/N$ where there are absorbing barriers. The Langevin equation (Stratonovich) becomes

$$\dot{m} = \frac{a_0}{2}m(1-m^2) + \frac{3\sigma}{2}\sqrt{K}m(1-m^2)\Xi(t), \tag{4.58}$$

where $\Xi(t)$ is a white Gaussian noise, $K = \mathcal{T}/3$ and $a_0 = 1 - 3b_0/2$.

At this point, it worths to make a change of variables since the integration limit $m = \pm 1 \mp 2/N$ is not accurate to work with. In this sense, one should consider the density of up spins (ρ) instead of the magnetization. The change is given by

$$\rho = \frac{m+1}{2}, \tag{4.59}$$

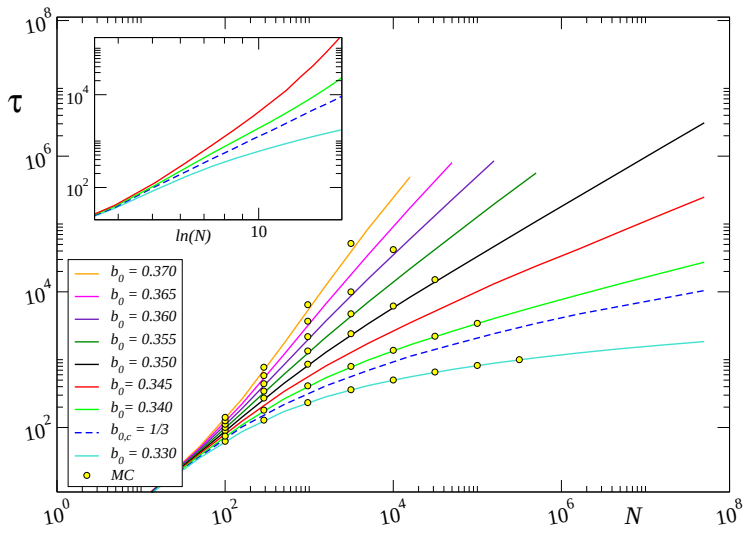

Figure 4.14. Main: Log-log plot of the mean lifetime as a function of the size of the system N. Monte Carlo simulations on a FCN (dots) and numerical integration of the langevin equation (4.57) for values of b going from 0.330 (bottom) to 0.370 (top) and $\sigma = 0.3$. In the active phase a finite region with power law scaling is observed. Inset: log-log plot of τ as a funtion of N. At the critical point it scales as $\tau \sim [\ln N]^{3.68}$.

so when the system is disordered $\rho = 1/2$, while $\rho = 0, 1$ in the absorbing states. Because of the symmetry of the problem it is possible to consider only one part of it; that is, define the extinction time as that need for reaching $\rho = 0$ from $\rho = 1/2$. Now, all the calculations are very similar to those of the Ising model presented in Section 4.4.1 and solved in Appendix B but with redefined parameters.

As the Stratonovich scheme is being used, usual chain rule works for the change of variables. The transformed equation is

$$\dot{\rho} = A(\rho) + \sqrt{K}C(\rho)\Xi(t), \tag{4.60}$$

with

$$\begin{aligned} A(\rho) &= a_0\rho(2\rho-1)(1-\rho) \\ C(\rho) &= 3\sigma\rho(2\rho-1)(1-\rho). \end{aligned} \tag{4.61}$$

From here, the steps are the same as in the Ising Model (Appendix B), though taking the initial point in $\rho = 1/2$ instead of in $m = 1$ as it was in the Ising model. It is not important because it does not affect the time asymptotic behavior in the limit of N large. It is

$$T \sim \begin{cases} \frac{\ln N}{6(b_0-1/3)} & \text{for } b_0 < 1/3 \\ \frac{(\ln N)^2}{3\mathcal{T}\sigma^2} & \text{for } b_0 = 1/3 \\ N^{\frac{2(b_0-1/3)}{\mathcal{T}\sigma^2}} & \text{for } b_0 > 1/3. \end{cases} \tag{4.62}$$

4.5.2 Finite dimension.

Simulations have been also run on a regular lattice with interactions to nearest neighbors in low dimensions. In $2d$ evidences of the existence of Temporal Griffiths Phases are not observed. The extinction time scales at the critical point as it does in the pure Voter model, $\tau \sim N\ln N$, and there is not a slow scaling (power law) broad region in the ferromagnetic phase (Figure 4.15).

On the other hand, regarding $d = 3$, TGPs do not seem to appear neither. The ordering time scales as it does in the three dimensional pure voter model ($\tau \sim N$), (Figure 4.16). In this case the critical point has not been calculated, but it lies between $(0.33, 0.34)$ when $\sigma = 0.3$, so the shifting in case of taking place is smaller than in 2d. This is what expected from previous results. The critical point shifts when external noise is introduced in low dimensions, but it tends to its value in the pure model in the limit $d \to \infty$ (mean-field).

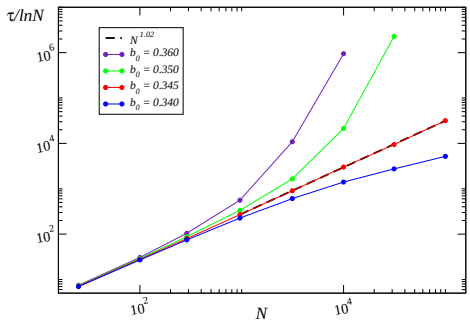

Figure 4.15. Log-log plot of the escape time $\tau/\ln N$ as a function of the number of spins N for the time dependent q-voter model $d = 2$ with $\sigma = 0.3$. Monte Carlo simulations on a regular lattice with nearest neighbors interactions. The behavior is similar to the observed in the pure model with a critical scaling $\tau \sim N\ln N$.From bottom to top $b_0 = 0.340, 0.345, 0.350, 0.360$.

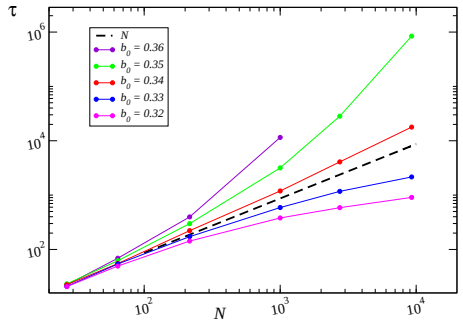

Figure 4.16. Log-log plot of the escape time $\tau/\ln N$ as a function of the number of spins N for the time dependent q-voter model $d = 3$ with $\sigma = 0.3$. Monte Carlo simulations on a regular lattice with nearest neighbors interactions. The behavior is similar to the observed in the pure model with a lineal scaling at criticality ($\tau \sim N$). From bottom to top $b_0 = 0.32, 0.33, 0.34, 0.35, 0.36$.

5

Summary and conclusions.

In this Master Thesis, the effect of temporal disorder around every phase transition presented in systems with two symmetric absorbing states has been investigated (Directed Percolation, Ising and Generalized Voter). The way we implemented the disorder was by varying the control parameter b of the model, randomly around a mean value b_0. At each Monte Carlo step a random number from a flat distribution of width 2σ was sorted and added to b_0. Working this way, and choosing σ properly, one makes the system wander between the active phase and the absorbing phase[1].

Above a critical dimension, the behavior of the system under this time varying external condition is very different to the corresponding behavior in the pure system, that is, without external forcing. The main features are:

1. A region appearing in the active phase such that the escape time scales as a power law of the system size N.

2. Logarithmic scaling with N of the mean first-passage time at criticality.

3. Generic divergences of magnitudes such as the susceptibility.

Because of its phenomenological similarities with the Griffiths Phases observed in systems with quenched disorder, the subregion appearing in the phase diagram has been called Temporal Griffiths Phases. These analogies and differences can be summarized as:

1. In GPs the disorder is "quenched in space" while in TGP it is "quenched in time".

[1] In the case of the Ising model considered in the Section 4.4 there are not absorbing states but an order-disorder phase transition has to be considered. The system shifts between tending to order or disorder.

2. The presence of quenched in space noise shifts the critical point. When considering temporal disorder it is also shifted, and the displacement is bigger the lower the dimension is. It approaches to the pure model value in the mean-field limit ($d \to \infty$).

3. In GPs active regions exist even if the state is absorbing. In TGPs absorbing temporal regions (time intervals where the system tends to the absorbing state) exist even if the state is active.

4. In GPs the probability of observing a rare region of size s is proportional to $e^{-\alpha s}$, where α is the disorder dependent constant. In TGPs a time interval of length T is occurs with probability $e^{-\kappa T}$, where κ is a disorder dependent constant too.

5. In GPs it is observed a generic power law decay in time of magnitudes such as density of particles. In TGPs the time needed by the system for reaching the absorbing state scales as a power-law with the size of the system. The critical scaling is not a power law anymore, but logarithmic. All exponents are not universal since they depend on the noise strength, decreasing upon increasing the noise amplitude.

Therefore, TGPs are analogous to GPs by exchanging the roles of space and time.

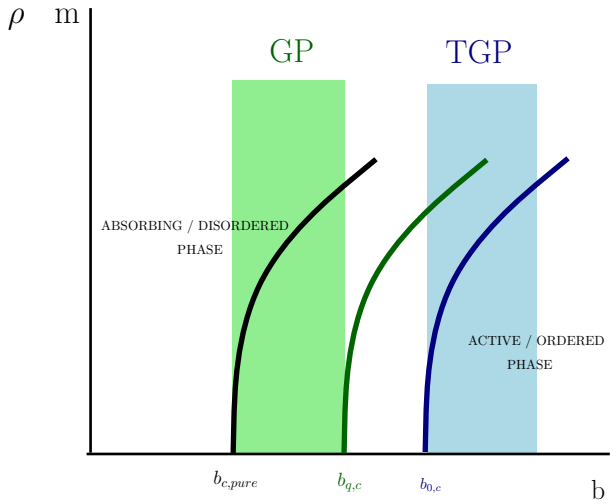

Figure 5.1. Sketch of the pase diagram for the pure (black), the quenched (green) and the temporal disorder (blue) models in the Ising and DP phase transition. Note that for the Ising transition $b \sim T^{-1}$.

The research has been driven through the framework provided by systems with two symmetric absorbing states trying to extend the work of Vázquez et al [39] to many other phase transitions. Due to the rich phenomenology shown by systems with Z_2 symmetry, the study of the different types of phase transitions that appear

in these systems, allowed us to reach some conclusions about the generic conditions for the existence of TGPs:

1. The Ising model presents a phase transition like the one occuring when a system with two absorbing states breaks its symmetry (Figure 4.2). However, it is the most representative model of many interacting particle systems relaxing to the thermodynamic equilibrium, and it does not have any absorbing state. Therefore, we argue that absorbing states are not relevant for the existence of TGPs.

2. The phase transition belonging to the Generalized Voter universality class is not a continuous but a first order phase transition (the order parameter presents a discontinuity at the critical point). TGPs also seem to appear in first order phase transitions.

Finally, one question is still open after this work. This is about the critical dimension for observing TGPs. It is shown to depend on the model and seems to be related to how fast the pure model reaches the absorbing state (or the disorder state), when it is favored by the dynamics of the system.

A

Appendix 1: Itô-Stratonovich discussion.

During this memory, Langevin equations have appeared many times. As was indicated in Section 2.3.3, the integration of stochastic differential equations with multiplicative white noise presents some problems because the integral of the noise is not well defined. These problems are solved choosing either the Itô or the Stratonovich definition of the integral.[1]

One or the other have been chosen all this work long depending on the origin of the noise term in the Langevin equation. In this appendix how Itô's calculus works, and the connection between Itô and Stratonovich schemes is also explained. It ends with a short discussion on when the Itô scheme has to be used and when the Stratonovich one.

A.1 Stochastic integration.

The aim at this point is giving a precise definition of the second integral in

$$M(t) - M(0) = \int_0^t f[M(s), s]ds + \int_0^t g[M(s), s]dW(s), \tag{A.1}$$

$$\int_0^t G(s)dW(s). \tag{A.2}$$

The integration interval $[0, t]$ is divided into n subintervals,

$$0 \leq t_1 \leq t_2 \leq t_3 ... \leq t_{n-1} \leq t_n, \tag{A.3}$$

[1] Any definition can be chosen or even made, but these two are the most often used.

and the intermediate points in each interval τ_i defined

$$\tau_i = t_{i-1} + \alpha(t_i - t_{i-1}). \tag{A.4}$$

The stochastic integral (A.2) is defined as the limit of the partial sums,

$$S_n = \sum_{i=1}^{n} G(\tau_i)(W(t_i) - W(t_{i-1})), \tag{A.5}$$

where the Itô vs Stratonovich dilemma resides in the fact that the limit of S_n depends on the particular set of points τ_i that are used. Itô stochastic integral is defined taking $\alpha = 0$, so (A.5) becomes

$$S_n = \sum_{i=1}^{n} G(t_{i-1})(W(t_i) - W(t_{i-1})), \tag{A.6}$$

that is, the known function $g(x(t))$ is evaluted on the beginning point of the interval while Stratonovich is obtained if $\alpha = 1/2$ and

$$S_n = \sum_{i=1}^{n} G\left(\frac{t_{i-1} + t_i}{2}\right)(W(t_i) - W(t_{i-1})). \tag{A.7}$$

A.2 Itô's formula.

In spite of being much more elegant from a mathematical point of view, Ito's prescription is not always the most suitable choice for physical interpretation. Calculus we are used to does not work in this scheme and a different change of variables must be considered. To begin with, lets take an arbitrary function $a[x(t)]$ with $x(t)$ obeying the SDE (2.13) and focus on

$$\begin{aligned} da[x(t)] &= a[x(t) + dx(t)] - a[x(t)] \\ &= a'[x(t)]dx(t) + \frac{1}{2}a''[x(t)]dx^2(t) + ... \\ &= a'[x(t)]\left\{f(x,t) + g(x,t)\xi(t)\right\}dt + \frac{1}{2}a''[x(t)]g^2(x,t)dW^2(t) + ..., \end{aligned} \tag{A.8}$$

where higher terms in dt have been neglected. Now, replacing $dW^2(t) = dt$ (see [12] for a proof),

$$da[x(t)] = a'[x(t)]\left\{f(x,t) + \frac{1}{2}a''[x(t)]g^2(x,t)\right\}dt + a''[x(t)]g(x,t)dW(t), \tag{A.9}$$

which is known as the Itô's formula and shows that change of variables is not given by ordinary calculus unless $a[x(t)]$ is linear in $x(t)$.

A.3 From Stratonovich to Itô.

As may be expected, both interpretations of the stochastic integral are somehow related. To show it, consider an stochastic differential equation

$$\frac{dx}{dt} = \alpha[x(t), t] + \beta[x(t), t]\eta(t), \tag{A.10}$$

where $\eta(t)$ is a white, zero mean, Gaussian noise. Integrating, it is,

$$x(t) = x(0) + \int_0^t \alpha[x(s), s]ds + S\int_0^t \beta[x(s), s]dW(s), \tag{A.11}$$

where S denotes that a Stratonovich integration is used. We will derive the equivalent Itô stochastic differential equation.

Assuming that $x(t)$ is a solution of

$$dx(t) = a[x(t), t]dt + b[x(t), t]dW(t), \tag{A.12}$$

the corresponding $\alpha[x(t), t]$ and $\beta[x(t), t]$ will be deduced. The first step is to compute the connection between $S\int_0^t \beta[x(s), s]dW(s)$ and $\int_0^t \beta[x(s), s]dW(s)$, where the lack of notation in the second integral means an Itô interpretation. Then,

$$S\int_0^t \beta[x(s), s]dW(s) \approx \sum_i \beta\left[\frac{x(t_i) + x(t_{i-1})}{2}, t_{i-1}\right][W(t_i) - W(t_{i-1})]\,. \tag{A.13}$$

Taking into an account

$$x(t_i) = x(t_{i-1}) + dx(t_{i-1}), \tag{A.14}$$

in the Stratonovich integral, then

$$\beta\left[\frac{x(t_i) + x(t_{i-1})}{2}, t_{i-1}\right] = \beta\left[x(t_{i-1}) + \frac{1}{2}dx(t_{i-1}), t_{i-1}\right]. \tag{A.15}$$

Now, the Itô SDE (A.12) is used in order to write

$$dx(t_i) = a[x(t_{i-1}), t_{i-1}](t_i - t_{i-1}) + b[x(t_{i-1}), t_{i-1}][W(t_i) - W(t_{i-1})]. \tag{A.16}$$

Using Itô's formula (A.9) as well as simplifying the notation writing $\beta(t_{i-1})$ instead of $\beta[x(t_{i-1}), t_{i-1}]$, (A.15) becomes,

$$\begin{aligned}\beta\left[\frac{x(t_i) + x(t_{i-1})}{2}, t_{i-1}\right] &= \beta(t_{i-1}) + \left[a(t_{i-1})\partial_x\beta(t_{i-1}) + \frac{1}{4}b^2(t_{i-1})\right]\left[\frac{1}{2}(t_i - t_{i-1})\right] + \\ &+ \frac{1}{2}b(t_{i-1})\partial_x\beta(t_{i-1})[W(t_i) - W(t_{i-1})].\end{aligned} \tag{A.17}$$

Finally, substituing into (A.13), neglecting terms in dt^2 and $dWdt$ and setting $dW^2 = dt$,

$$S\int = \sum_i \beta(t_{i-1})(W(t_i) - W(t_{i-1})) + \frac{1}{2}\sum_i b(t_{i-1})\partial_x\beta(t_{i-1})(t_i - t_{i-1}), \quad \text{(A.18)}$$

or going back to integrals,

$$S\int_0^t \beta[x(s),s]dW(s) = \int_0^t \beta[x(s),s]dW(s) + \frac{1}{2}\int_0^t b[x(s),s]\partial_x\beta[x(s),s]ds, \quad \text{(A.19)}$$

which means that the stochastic integral in Stratonovich representation is equivalent to a stochastic integral in Itô's and a drift term. It is also important to remark that this formula gives a connection between both integrals of function $\beta[x(s),s]$, in which $x(s)$ is the solution of the Itô SDE (A.12). It does not give a general connection between the Itô and Stratonovich integrals of arbitrary functions.

$$\begin{aligned} \text{The Itô SDE} \qquad & dx = a(x,t)dt + b(x,t)dW(t) \\ \text{is the Stratonovich SDE} \quad & dx = \left[a(x,t) - \frac{1}{2}b(x,t)\partial_x b(x,t)\right]dt + b(x,t)dW(t). \end{aligned} \quad \text{(A.20)}$$

Or

$$\begin{aligned} \text{The Stratonovich SDE} \quad & dx = \alpha dt + \beta dW(t) \\ \text{is the Itô SDE} \qquad & dx = \left[\alpha(x,t) + \frac{1}{2}\beta(x,t)\partial_x\beta(x,t)\right]dt + \beta(x,t)dW(t). \end{aligned} \quad \text{(A.21)}$$

There are many consequences of this transformation formula, but the more important are

- It is always possible to change from the Stratonovich to the Itô interpretation of a SDE by adding $\frac{1}{2}\beta(x,t)\partial_x\beta(x,t)$ or in the inverse direction subtracting a similar term.
- In the case of additive noise, i.e., $g(x,t) = const.$ in (2.13) there is no difference between the Itô and Stratonovich integral.
- In the case of multiplicative noise, i.e., $g(x,t) \neq const.$ in (2.13), where the influence of the random force depends on the state of the process, the correlation between both the random force and the state of the process is implicit in the Stratonovich integral. It gives raise to the noise induced drift when moving to Itô appearing in the deterministic part of the equation.

- The Stratonovich calculus obeys the classical chain rule, Itô's formula derived in Section A.2 plays a similar role on Itô's calculus.

A.4 Stratonovich / Itô dilemma.

The long controversy in the physical literature about what is the right definition of the stochastic integral has created some confusion on this topic. That's why, although a much more mathematically rigorous and longer discussion can be found in the references, [11, 43, 49] some hand waving arguments will be given in this section.

First of all it is important to say that this kind of ambiguity when working with SDE only yields for the particular, but most common, case of differential equations with multiplicative white noise[2]. As a first approach, it is natural to tend to believe that due to invariance of the equations under "coordinate transformation" $y = u(x)$ when working on Stratonovich scheme it is the proper choice. However, it means nothing but it obeys the classical calculus rules we are familiar with. The only quantities that have to be invariant under a transformation $u = y(x)$, where u is one to one, are the probabilities,

$$p(y,t)dy = p(x,t)dx, \tag{A.22}$$

and this is of course guaranteed in both calculi. They lead to a consistent calculus.

It looks sensible, then, to change the question. The matter is not what is the right definition of the stochastic integral, but how do we model real systems by stochastic processes. That is, in which situation either Itô's or Stratonovich's choice is the most suitable.

On the one hand, if the starting point is a phenomenological equation in which some fluctuating parameters represented through colored noise terms are approximated by Gaussian white noise, then the most appropiate process is the one that is defined by the Stratonovich interpretation of the equation.

On the other hand, in many systems the appropiate starting point is a discret time equation, as it happens, for instance, in biology when working with populations of insects. In these cases the equation reads

$$X(t_i) = X(t_{i-1}) + f(X(t_{i-1}))\Delta t + \sigma g(X(t_{i-1}))Q(t_{i-1}), \tag{A.23}$$

where $t_i = t_{i-1} + \Delta t$ in every time step and Q_i are Gaussian independent random variables with expected values $< Q(t_i) >= 0$ and $< Q^2(t_i) >= \Delta t$.

[2]Cases where the rapidly fluctuating external force depends on the state of the system.

If times considered are longer compared to Δt, the continuous time limit can be taken. Then the system is described by

$$\dot{X}(t) = f[X(t)] + \sigma g[X(t)]\dot{W}(t), \tag{A.24}$$

which is also a SDE where $W(t)$ is the Wiener process. However, due to the asymmetric form of (A.23) with respect to time it is much more appropiate the stochastic process defined according to the Itô interpretation in this case.

To sum up, as a take to home message from this section, two different cases can be considered when working with SDE. When the white gaussian noise limit is considered as the limit of a colored noise when the correlation time tends to zero, the Stratonovich interpretation is more sensible, when Itô's is more suitable when it represents the continuous limit of a discrete time problem. In any case, there are no universally valid theoretical reasons why one or the other interpretation of an SDE should be preferred and the ultimate test must be the confrontation of the analytical (or numerical) results with the experimental facts.

B

Appendix 2: Analytical calculations on the escape time for the Ising Model.

Here, all the analytical calculations done to obtain the result (4.52) are shown in detail. Taking into account that it is taken $g_{1,0}(m) = 0$ to make integrals analytically solvable the Langevin equation is

$$\dot{m} = f(m) + \sqrt{D} g_2(m) \Xi(t), \tag{B.1}$$

with

$$\begin{aligned} f(m) &= (b_0 - 1)m - \frac{(b_0 m)^3}{3} = a_0 m - c_0 m^3, \\ g_2(m) &= \sigma m (1 - b_0^2 m^2), \end{aligned} \tag{B.2}$$

and the white gaussian noise defined by its statistical properties

$$\begin{aligned} < \Xi(t)\Xi(t') > &= \delta(t - t'), \\ < \Xi(t) > &= 0. \end{aligned} \tag{B.3}$$

It is easy to see how the Langevin equation (B.1) presents one absorbing state in $m = 0$ appearing induced by the simplification done neglecting thermal fluctuations.

Working in the Stratonovich scheme [1], the associated Fokker-Planck equation is

$$\begin{aligned} \frac{\partial P(m,t)}{\partial t} &= -\frac{\partial}{\partial m}\left[f(m) + \frac{D}{2} g_2(m) g_2'(m) \right] P(m,t) \\ &+ \frac{D}{2}\frac{\partial^2}{\partial m^2}\left[g_2^2(m) P(m,t) \right], \end{aligned} \tag{B.4}$$

[1] It has to be in that way because the noise term comes from taking the white noise limit in a colored one

where

$$f(m)+\frac{D}{2}g_2(m)g_2'(m) = \left(a_0+\frac{\tau}{6}\sigma^2\right)m+\frac{\tau\sigma^2}{2}b_0^4m^5 - \\ - \left(c_0+\frac{2\tau\sigma^2b_0^2}{3}\right)m^3, \tag{B.5}$$

$$Dg_2^2(m) = \frac{\tau}{3}\sigma^2m^2\left(1-b_0^2m^2\right)^2. \tag{B.6}$$

According to [12], the escape time from an starting point m obeys,

$$\left[f(m)+\frac{D}{2}g_2(m)g_2'(m)\right]T'(m)+\frac{1}{2}Dg_2^2(m)T''(m)=-1. \tag{B.7}$$

As the size of the system does not appear naturally in the problem because of the simplification done when taking $g_{1,0}=0$, the mean escape time will be defined as that needed to pass through $m=2/N$, which is the length of the jumps of the brownian particle to whose movement the problem has been mapped. Then, taking into an account that there is an absorbing barrier in $m=0$ and a reflecting one in $m=1$ and the initial condition, the solution is [12]

$$T(m_i=1)=2\int_{2/N}^{m_i=1}\frac{dy}{\psi(y)}\int_y^1\frac{\psi(z)}{Dg_2^2(z)}dz, \tag{B.8}$$

with

$$\psi(z)=\exp\int_{2/N}^{z}dz'\frac{2f(z')+Dg_2(z')g_2'(z')}{Dg_2^2(z')}, \tag{B.9}$$

which involves 6^{th} and 4^{th} order polynomial functions.

In order to make the integral simpler two assumptions are now done. One the one hand the functions are expanded up to 3^{rd} order, and on the other hand the lower integration limit in (B.9) is taken as 1 instead of $2/N$. This change can be done because $\psi(z)$ appears both in the numerator and the denominator of $T(m)$, so the contribution of the lower limit vanishes allowing to take it in our interest. The first assumption leads to

$$f(m)+\frac{D}{2}g_2(m)g_2'(m) \approx m(r-sm^2), \\ Dg_2^2(m) \approx \omega m^2, \tag{B.10}$$

where it has been defined $\omega\equiv\tau\sigma^2/3$; $r\equiv a_0+\omega/2$; $s\equiv(c_0+2\omega b_0^2)$. The size of the system will be rescaled too, so the lower limit in the expression of the escape time (B.8) is $1/N$. This simplifies the notation and does not affect the qualitative behaviour of the results in the asymptotic limit (only a constant factor appears).

Now, it can be written,

$$\psi(z) = \exp \int_1^z \frac{2z'(r - sz'^2)}{\omega z'^2} dz' = z^\alpha \mathrm{e}^{\beta(1-z^2)}, \tag{B.11}$$

where $\alpha \equiv 2r/\omega$ and $\beta \equiv s/\omega$.

Lets now define the function

$$I(y) = \int_y^1 \frac{\psi(z)}{Dg_2^2(z)} dz = \frac{\mathrm{e}^\beta}{\omega} \int_y^1 z^{\alpha-2} \mathrm{e}^{-\beta z^2} dz, \tag{B.12}$$

which presents a singularity when $\alpha = 1$ as can be seen integrating by parts. With the definition made of the parameters, it can be shown that it corresponds to $b_0 = 1 \equiv b_{0,c}$.

Considering definitions (B.11) and (B.12), the mean escape time is given by

$$T = 2 \int_{1/N}^1 \frac{I(y)}{\psi(y)} dy. \tag{B.13}$$

Each case will be studied separately.

B.1 Case $\alpha \neq 1$.

Integrating by parts (B.12)

$$I(y) = \frac{\mathrm{e}^\beta}{\omega} \left[\frac{\mathrm{e}^{-\beta} - \mathrm{e}^{-\beta y^2} y^{\alpha-1}}{\alpha - 1} + 2\beta \int_y^1 \frac{z^\alpha \mathrm{e}^{-\beta z^2}}{\alpha - 1} dz \right], \tag{B.14}$$

where the new integral can be solved again integrating by parts. Working recursively this way,

$$I(y) = \frac{\mathrm{e}^\beta}{\omega} \left[\frac{\mathrm{e}^{-\beta} - \mathrm{e}^{-\beta y^2} y^{\alpha-1}}{\alpha - 1} + 2\beta \frac{\mathrm{e}^{-\beta} - \mathrm{e}^{-\beta y^2} y^{\alpha+1}}{(\alpha-1)(\alpha+1)} + ... \right], \tag{B.15}$$

or

$$I(y) = \frac{1}{\omega} \sum_{k=0}^\infty (2\beta)^k \frac{1 - \mathrm{e}^{-\beta(y^2-1)} y^{\alpha-1+2k}}{\prod_{i=0}^k (\alpha - 1 + 2i)}. \tag{B.16}$$

The mean escape time is given now by

$$T = \frac{2}{\omega} \sum_{k=0}^\infty \frac{(2\beta)^k}{\prod_{i=0}^k (\alpha - 1 + 2i} \left[I_1(N) - I_2(k, N) \right], \tag{B.17}$$

where

$$I_1(N) \equiv \int_{1/N}^{1} y^{-\alpha} \mathrm{e}^{\beta(y^2-1)} dy \tag{B.18}$$

$$I_2(k,N) \equiv \int_{1/N}^{1} y^{2k-1} dy. \tag{B.19}$$

Integrating $I_1(N)$ by parts (taking again the exponential part as u and the rest as dv) and following the same procedure as in (B.12) it is obtained

$$I_1(N) = \sum_{l=0}^{\infty} \frac{(-2\beta)^l [1 - N^{\alpha-1-2l} \mathrm{e}^{\beta(1/N^2-1)}]}{\prod_{j=0}^{l}(\alpha - 1 + 2j)}, \tag{B.20}$$

while $I_2(k,N)$ is easily solved

$$I_2(k,N) = \begin{cases} -\ln(N^{-1}) = \ln(N) & \text{for } k = 0, \\ \frac{1-N^{-2k}}{2k} & \text{for } k \geq 1. \end{cases} \tag{B.21}$$

At the end, an expresion for the mean escape time is achieved

$$\begin{aligned} T &= \frac{2}{\omega}\left(\frac{I_1(N) - \ln(N)}{\alpha - 1}\right) + \\ &+ \frac{2}{\omega}\sum_{k=1}^{\infty} \frac{(2\beta)^k \left[I_1(N) - (1 - N^{-2k})/2k\right]}{\prod_{i=0}^{k}(\alpha - 1 + 2i)}. \end{aligned} \tag{B.22}$$

In the asymptotic limit $N \to \infty$ two different cases must be considered.

B.1.1 $\alpha < 1$

Under this prescription, $\alpha - 1 - 2l < 0$ when $l \geq 0$ so in $I_1(N)$

$$1 - N^{\alpha-1-2l} \mathrm{e}^{\beta(1/N^2-1)} \sim 1 - \frac{\mathrm{e}^{-\beta}}{N^{\nu}} \sim 1, \tag{B.23}$$

which leads to

$$I_1(N) = \sum_{l=0}^{\infty} \frac{(-2\beta)^l}{\prod_{j=0}^{l}(1 + 2j - \alpha} \equiv C(\alpha, \beta). \tag{B.24}$$

Finally, for the mean escape time,

$$T \approx \frac{2}{\omega}\left[\frac{C(\alpha,\beta) - \ln(N)}{\alpha - 1} + \sum_{k=1}^{\infty} \frac{(2\beta)^k \left(C(\alpha,\beta) - (2k)^{-1}\right)}{\prod_{j=0}^{l}(\alpha - 1 + 2j)}\right], \tag{B.25}$$

what means,

$$T \approx \frac{2}{\omega(\alpha - 1)} \ln(N). \tag{B.26}$$

B.1.2 $\alpha > 1$

It is taken as a starting point

$$I_1(N) = \sum_{l=0}^{\infty} \frac{(-2\beta)^l [1 - N^{\alpha-1-2l} \mathrm{e}^{\beta(1/N^2-1)}]}{\prod_{j=0}^{l}(\alpha - 1 + 2j)}, \tag{B.27}$$

where considering that $N^{\alpha-1} \gg N^{\alpha-1-2l}, \forall l > 0$, only the first term in (B.27) is relevant. It implies

$$I_1(N) \approx \frac{1 - \mathrm{e}^{-\beta} N^{\alpha-1}}{1-\alpha} \approx \frac{\mathrm{e}^{-\beta} N^{\alpha-1}}{1-\alpha}, \tag{B.28}$$

and in the mean escape time

$$T \approx K(\alpha,\beta) N^{\alpha-1} - \frac{2\ln(N)}{\omega(\alpha-1)} \sim N^{\alpha-1} \quad (N \gg 1). \tag{B.29}$$

B.2 Case $\alpha = 1$. Critical point.

It has to be solved now

$$I(y) = \int_y^1 \frac{\psi(z)}{D g_2^2(z)} dz = \frac{\mathrm{e}^{\beta}}{\omega} \int_y^1 y^{-1} \mathrm{e}^{-\beta z^2} dz, \tag{B.30}$$

using the expansion of the exponential function and integrating it is

$$I(y) = \frac{\mathrm{e}^{\beta}}{\omega} \left[-\ln(y) + \sum_{k=1}^{\infty} \frac{(-\beta)^k (1-2y)^{2k}}{k! 2k} \right]. \tag{B.31}$$

It makes the mean escape time to obey, taking the form of $I(y)$ Eq.(B.31) into Eq.(B.13)

$$T = \frac{2\mathrm{e}^{\beta}}{\omega} \left[I_3(N) + \sum_{k=1}^{\infty} \frac{(-\beta)^k}{k! 2k} \left(I_4(N) + I_5(k,N) \right) \right], \tag{B.32}$$

where

$$\begin{aligned} I_3(N) &= -\int_{1/N}^{1} \ln(y) y^{-1} \mathrm{e}^{\beta(y^2-1)} dy, \\ I_4(N) &= \int_{1/N}^{1} y^{-1} \mathrm{e}^{\beta(y^2-1)} dy, \\ I_5(k,N) &= \int_{1/N}^{1} y^{k-1} \mathrm{e}^{\beta(y^2-1)} dy. \end{aligned} \tag{B.33}$$

First of all, lets consider the solution of $I_3(N)$ integrating by parts ($u = \mathrm{e}^{\beta y^2}$ and $dv = \ln(y)y^{-1}dy$) so,

$$I_3(N) = \frac{(\ln N)^2}{2}\mathrm{e}^{\beta(N^{-2}-1)} + \beta\int_{1/N}^{1}(\ln y)^2\mathrm{e}^{\beta(y^2-1)}dy, \tag{B.34}$$

where the new integral is solved again integrating by parts taking

$$\begin{aligned} u &= \mathrm{e}^{\beta(y^2-1)} \rightarrow du = 2\beta\mathrm{e}^{\beta(y^2-1)} \\ dv &= (\ln y)^2 dy \rightarrow v = 2y - 2y\ln y + y(\ln y)^2. \end{aligned} \tag{B.35}$$

It leads to a solution behaving like

$$\beta\int_{1/N}^{1}(\ln y)^2\mathrm{e}^{\beta(y^2-1)}dy = 2\beta - O(N^{-1}) + O\left(\frac{\ln N}{N}\right), \tag{B.36}$$

so finally,

$$I_3(N) = \frac{(\ln N)^2}{2}\mathrm{e}^{\beta(N^{-2}-1)} + 2\beta - O(N^{-1}) + O\left(\frac{\ln N}{N}\right), \tag{B.37}$$

which scales in the asymptotic limit as

$$I_3(N) \sim \frac{(\ln N)^2}{2}\mathrm{e}^{-\beta}. \tag{B.38}$$

Secondly, lets focus on $I_4(N)$, where, again, an expansion of the exponential function has to be done

$$I_4(N) = \int_{1/N}^{1} y^{-1}\mathrm{e}^{\beta(y^2-1)}dy = \mathrm{e}^{-\beta}\int_{1/N}^{1} y^{-1}\sum_{k=0}^{\infty}\frac{\beta^k y^{2k}}{k!}dy \tag{B.39}$$

which can be easily solved

$$I_4(N) = \mathrm{e}^{-\beta}\left[\ln N + \frac{\beta^k}{k!2k}\left(1 - N^{-2k}\right)\right]. \tag{B.40}$$

The leading behavior when the size of the system is big enough ($N \gg 1$) is

$$I_4(N) \sim \mathrm{e}^{-\beta}\ln N + C_4(\beta). \tag{B.41}$$

The last integral to be solved, also using the expansion of the exponential function, is

$$\begin{aligned} I_5(k,N) = \mathrm{e}^{-\beta}\sum_{l=0}^{\infty}\frac{\beta^l}{l!(k+2l)}\left(1 - N^{-2l-k}\right) &\sim cte \\ N &\gg 1. \end{aligned} \tag{B.42}$$

It finally leads to an expression for the mean escape time in the critical point

$$T \approx \frac{2\mathrm{e}^{-\beta}}{\omega}\left\{\frac{\mathrm{e}^{-\beta}(\ln N)^2}{2} + \sum_{k=1}^{\infty}\frac{(-\beta)^k}{k!2k}\left[\mathrm{e}^{-\beta}\ln N + C_4'(\beta)\right]\right\}. \tag{B.43}$$

In the limit of very big systems ($N \gg 1$) the mean escape time scales as

$$T \sim \frac{(\ln N)^2}{\omega} + \frac{1}{\omega}\sum_{k=1}^{\infty}\frac{(-\beta)^k}{k!k}\ln N + K(\beta), \tag{B.44}$$

with asymptotic behaviour

$$T \sim \frac{(\ln N)^2}{\omega}. \tag{B.45}$$

To sum up, it has been obtained analytically the finite size scaling of the mean escape time, defined as the time taken by the system for reaching $m = 0$ from an initial condition $m_i = 1$. It is

$$T \sim \begin{cases} \frac{2}{\omega(\alpha-1)}\ln N & \text{for } \alpha < 1, \\ \frac{(\ln N)^2}{\omega} & \text{for } \alpha = 1, \\ N^{\alpha-1} & \text{for } \alpha > 1. \end{cases} \tag{B.46}$$

or in terms of the original parameters

$$T \sim \begin{cases} \frac{\ln N}{b_0-1} & \text{for } b_0 < b_{0,c}, \\ \frac{3(\ln N)^2}{\tau\sigma^2} & \text{for } b_0 = b_{0,c}, \\ N^{\frac{6(b_0-1)}{\tau\sigma^2}} & \text{for } b_0 > b_{0,c}. \end{cases} \tag{B.47}$$

Bibliography

[1] H. J. Jensen, E. Arcaute. Annals of the New York Academy of Science **1195**, E19-E26, (2010).

[2] S. E. Jørgensen *et al. A New Ecology. Systems Perspective,* Elsevier, 2007.

[3] J. D. Murray. *Mathematical Biology. Vols I and II,* Springer, Berlin-Heidelberg, 2004.

[4] F. Vázquez, C. López, J. Calabrese and M.A. Muñoz. Journal of Theoretical Biology **264**, 360-366, (2010).

[5] E. G. Leigh Jr., J. Theor. Biol. **90**, 213 (1981).

[6] A. Kamenev, B. Meerson and B. Shklovskii, Phys. Rev. Lett. **101**, 268103 (2008).

[7] R. B. Griffiths, Phys. Rev. Lett. **23**, 17, (1969).

[8] T. Votja, J. Phys. A **39**, R143, (2006).

[9] T. M. Ligget, *Interacting particle systems*, Springer, Berlin, 1985.

[10] H. Hinrichsen. *Lectures held at the International Summer School on Problems in Statistical Physics XI,* Leuven, 2005.

[11] N. G. Van Kampen, *Stochastic Processes in Physics and Chemistry,* North-Holland, Amsterdam, 2004.

[12] C. W. Gardiner. *Handbook of Stochastic Methods: for Physics, Chemistry and the Natural Sciences,* Springer-Verlag, Heidelberg, 1985.

[13] H. Risken. *The Fokker Planck equation. Methods of Solution and Applications,* Springer-Verlag, 1988.

[14] J. M. Yeomans. *Statistical physics of phase transitions,* Oxford University Press, 1992.

[15] L. Kadanoff. *Statistitcal Physics. Statics, Dynamics and Renormalization,* World Scientific, Singapore, 2000.

[16] K. Huang. *Statistical Mechanics* John Wiley and Sons, 1987

[17] H. Hinrichsen, Adv. Phys. **49**, 815 (2000).

[18] H. Spohn, *Large scale dynamics of interacting particles*, Springer, Berlin, 1991.

[19] M. Henkel, *Conformal Invariance and Critical Phenomena,* Springer-Verlag, Berlin, 1999.

[20] M. San Miguel. *Stochastics Methods and Models in the Dynamics of Phase Transitions in Stochastic processes applied to physics* p.132-160. World Sci. Publ. 1985.

[21] P. Krapivsky, S. Redner and Eli Ben-Naim. *A Kinetic View of Statistical Physics.* Cambridge University Press, 2010.

[22] S. Redner. *A Guide to First-Passage Processes,* Cambridge University Press, New York, 2001.

[23] L. F. Lafuerza and R. Toral, Journal of Statistical Physics **140**, 917-933 (2010)

[24] D. T. Gillespie. J. Phys. Chem. **81**, 2340-2361, (1977).

[25] O. Al Hammal. *Langevin Equations for Nonequilibrium Phase Transitions.* (Ph.D Thesis), Granada, 2007.

[26] F. Mauricio Dominguez. *Fundamentos de Mecánica Estadística. Notas de Física Estadística I.* Universidad de La Laguna. La Laguna.

[27] U. C. Tuler *Field-theoretic Methods.*

[28] H. Park, AKPA Newsletter **18**, 83, (1997).

[29] M. Droz, A.L. Ferreira, A. Lipowski, Phys. Rev. E, **67**, 056108, (2003).

[30] F. Vázquez, C. López, Phys. Rev. E, **78**, 061127 (2008).

[31] O. Al Hammal, H. Chaté, I. Dornic and M.A. Muñoz, Phys. Rev. Lett. **94**, 230611, (2005).

[32] G. Ódor. *Universality in Nonequilibrium Models. Theoretical Foundations.* World Scientific, Singapore, 2008.

[33] J. Marro and R. Dickman. *Nonequilibrium phase transitions in lattice models.* Cambridge University Press, Cambridge, 1999.

[34] H. K. Janssen, Z. Phys. B. **42**, 151, (1981).

[35] P. Grassberger, Z. Phys. B. **47**, 365, (1982).

[36] G. Grinstein and A. Luther, Phys. Rev. B **13**, 1329, (1976).

[37] M. A. Muñoz, R. Juhsz, C. Castellano, and G. Ódor, Phys. Rev. Lett. **105**, 128701, (2010).

[38] N. Goldenfeld, *Lectures on Phase Transitions and the Renormalization Group*, Addison-Wesley, 1992.

[39] F. Vázquez, J. A. Bonachela, C. López, M. A. Muñoz, Phys. Rev. Lett. **106**, 235702, (2011).

[40] R. Martínez-García, F. Vázquez, C. López, M. A. Muñoz. (In preparation)

[41] I. Dornic, H. Chaté, J. Chavé, H. Hinrichsen, Phys. Rev. Lett. **87**, 045701, (2001).

[42] R. Dickman and A. Yu. Tretyakov, Phys. Rev. E, **52**, 3218 (1995).

[43] W. Horsthemke and R. Lefever *Noise-Induced Transitions. Theory and Applications in Physics, Chemistry and Biology,* Springer-Verlag, Heidelberg, 1984.

[44] G. Grinstein, M. A. Muñoz and Y. Tu. Phys. Rev. Lett. **76**, 4376, (1996).

[45] M. A. Muñoz. Multiplicative Noise in Non-Equilibrium Phase Transitions: A Tutorial. Advances in Condensed Matter and Statistical Mechanics Nova Science Publishers, 2004.

[46] Heuer, H.-O., Phys. A, **26**, L333-L339. (1993).

[47] J. J. Alonso and M. A Muñoz. European Phys. Lett., **56** (4), pp.485-491 (2001).

[48] C. Castellano, M. A. Muñoz an R. Pastor-Satorras. Phys. Rev. E **80**, 041129 (2009).

[49] A.H. Jazwinsky. *Stochastic Processes and Filtering Theory*. Academic, New York, 1970.